# Contents

1 Lecture 1: K3 surfaces: the definition and examples: Giacomo Cherubini (29/09) .................................................. 4
   1.1 Algebraic K3 surfaces .................................................. 4
      1.1.1 Appendix by Giacomo Cherubini .................................. 6
   1.2 Complex K3 surfaces .................................................. 9

2 Lecture 2: Topology of K3 surfaces: Annelies Jaspers (06/10) .......................................................... 10
   2.1 Singular cohomologies of complex K3 surfaces .......................................................... 10
   2.2 Simply connectedness .................................................. 11
   2.3 Appendix by Annelies Jaspers and Nadim Rustom .................................................. 12
      2.3.1 Complex K3 surfaces ................................................ 12
      2.3.2 Serre’s GAGA principle .............................................. 12
      2.3.3 Singular cohomology ............................................... 13
      2.3.4 Cup product ...................................................... 16
      2.3.5 Simply connectedness .............................................. 16

3 Lecture 3: Hodge structures: Simon Rose (20/10) .......................................................... 17
   3.1 Hodge structures ...................................................... 17
   3.2 Hodge structures of weight one ........................................ 18
   3.3 Hodge structures of weight two ........................................ 19
   3.4 Appendix by Simon Rose ................................................ 20
      3.4.1 Hodge Structures .................................................. 20
      3.4.2 The Hodge Diamond ............................................... 23
      3.4.3 Classifications .................................................... 28

4 Lecture 4: Torelli theorem for complex K3 surfaces: Nadim Rustom (27/10) .................................................. 30
   4.1 Torelli theorem: statements ............................................. 30
   4.2 Lattices and discriminant forms ......................................... 31
   4.3 Applications ...................................................... 32
CONTENTS

4.4 Appendix by Nadim Rustom ........................................... 32
  4.4.1 Néron-Severi lattice ........................................... 32
  4.4.2 Kähler cone and Positive cone ................................. 33
  4.4.3 Marked K3 surfaces ............................................ 35
  4.4.4 Torelli theorems .............................................. 36
  4.4.5 Surjectivity of the period map ................................ 37
  4.4.6 Lattices and discriminant forms ............................. 37
  4.4.7 Application ................................................... 38

5 Lecture 5: The weak and strong Torelli theorem: a proof 38
  5.1 Torelli theorem for Kummer surfaces: Dan Petersen (3/11) ...... 39
  5.2 Deformation theory, the local Torelli theorem for K3 surfaces: Dustin Clausen (10/11) ........................................ 40
  5.3 A proof of Torelli theorem for complex K3 surfaces: Lars Halvard Halle (18/11) .................................................. 42

6 Lecture 6: Surjectivity of the period mapping: a proof: Sho Tanimoto (25/11) 43
  6.1 Hausdorff reduction .............................................. 43
  6.2 Twistor lines .................................................... 44
  6.3 Hyperkähler structures on K3 surfaces .......................... 44
  6.4 A proof .......................................................... 45

7 Lecture 7: Moduli spaces of polarized K3 surfaces: Dan Petersen (2/12) 46
  7.1 Moduli functor .................................................. 46
  7.2 Hilbert schemes ............................................... 47
  7.3 Moduli spaces via the global period domain .................... 48

8 Lecture 8: Kodaira dimension of moduli spaces: Sho Tanimoto (9/12) 49
  8.1 Unirationality .................................................. 49
  8.2 Kodaira dimension ............................................ 49
  8.3 Low weight cusp form trick ................................. 50
  8.4 Borcherds form ............................................... 51

9 Lecture 9: K3 surfaces of geometric Picard rank 1 defined over number fields: Dino Destefano (11/2) 52
  9.1 Results by Ellenberg .......................................... 52

10 Lecture 10: Examples by van Luijk: Fabien Pazuki (16/2) 52
  10.1 Specializations, reduction modulo $p$ ........................................ 53
  10.2 The cycle class map ........................................... 54
### Introduction

This is a note for ANT reading seminar in the academic year 2015-2016 at Copenhagen. The main theme for this seminar is the geometry and arithmetic of K3 surfaces. Possible topics are

- K3 surfaces: the definition and examples
- Topology of K3 surfaces, lattice theory
- Hodge structures, periods, Torelli theorem
- Moduli spaces of polarized K3 surfaces
1 Lecture 1: K3 surfaces: the definition and examples: Giacomo Cherubini (29/09)

This lecture mainly follows Chapter 1 of [13]. Also see Section 1 of [30]. Let $k$ be an arbitrary field. A variety over $k$ is a geometrically integral separated scheme of finite type over $k$.

1.1 Algebraic K3 surfaces

We start our discussion from the definition of K3 surfaces.

**Definition** An algebraic K3 surface is a smooth projective 2-dimensional variety $X$ over $k$ such that $\omega_X \cong O_X$ and $H^1(X, O_X) = 0$.

There are many examples of K3 surfaces. Here are some examples.

**Example** (K3 surfaces of degrees 4, 6, and 8) Let $X$ be a smooth complete intersection of type $(d_1, \cdots, d_r)$ in $\mathbb{P}^n_k$. Without loss of generality, we may assume that $d_i \geq 2$. Then $X$ becomes a K3 surface only when

- $n = 3$ and $d_1 = 4$,
- $n = 4$ and $(d_1, d_2) = 2, 3$,
- $n = 5$ and $(d_1, d_2, d_3) = (2, 2, 2)$.
1.1 Algebraic K3 surfaces

Exercise 1.1. Show that these complete intersections are K3 surfaces.

Example (K3 surfaces of degree 2) Suppose that \( \mathrm{char} \ k \neq 2 \). Let \( X \) be a double cover of \( \mathbb{P}^2 \) branched along a smooth sextic curve \( C \) in \( \mathbb{P}^2 \). Then \( X \) is a K3 surface.

Exercise 1.2. Show that the above surface is a K3 surface.

Example (Kummer surfaces) Let \( k \) be a field of characteristic \( \neq 2 \). Suppose that \( A \) be an abelian surface over \( k \). Consider the following involution \( \iota: A \to A \) given by \( x \mapsto -x \). The fixed part of this involution consists of 16 \( \bar{k} \)-points. Let \( \rho: \tilde{A} \to A \) be the blow up of this \( k \)-scheme. The involution \( \iota \) lifts to the involution \( \tilde{\iota} \) on \( \tilde{A} \), and we consider the quotient \( \pi: \tilde{A} \to \tilde{A}/\tilde{\iota} = X \). This \( X \) is a K3 surface and it is called a Kummer surface.

Exercise 1.3. Show that the above surface is a K3 surface.

More non-standard examples are:

Example (K3 surfaces of degree 14) Consider the Plücker embedding \( \text{Gr}(2,6) \hookrightarrow \mathbb{P}^{14} \). This has degree 14. Then the intersection \( X = \text{Gr}(2,6) \cap \mathbb{P}^8 \) with a generic linear subspace is a K3 surface.

Mukai produced more examples of such descriptions of K3 surfaces of degree \( 2d = 2, 4, \cdots, 18 \). These descriptions can be used to prove that the moduli space of polarized K3 surfaces of low degree is unirational.

Exercise 1.4. Let \( X \) be an algebraic K3 surface. Show that \( \chi(X, O_X) = 2 \).

Theorem 1.5. (Riemann-Roch theorem) Let \( X \) be an algebraic K3 surface and \( L \) a line bundle on \( X \). Then
\[
\chi(X, L) = \frac{L^2}{2} + 2.
\]

Proposition 1.6. Let \( X \) be an algebraic K3 surface and \( C \) a connected one dimensional subscheme on \( X \). Then
\[
2p_a(C) - 2 = C^2.
\]
In particular, \( \text{NS}(X) \) is an even lattice.

Proposition 1.7. Let \( X \) be an algebraic K3 surface. Then natural surjections
\[
\text{Pic}(X) \to \text{NS}(X) \to \text{Num}(X)
\]
are isomorphisms. In other words, linear equivalence, algebraic equivalence, and numerical equivalence are all equivalent for divisors.

We call the rank of \( \text{NS}(X) \) by the Néron-Severi rank and denote it by \( \rho(X) \).

Theorem 1.8. (Hodge index theorem) Let \( X \) be an algebraic K3 surface. Then the signature of \( \text{NS}(X) \) is \( (1, \rho(X) - 1) \).
1.1.1 Appendix by Giacomo Cherubini

A very short note on Algebraic geometry and K3 surfaces
Giacomo Cherubini - May 27, 2016

Definition. A Variety $X$ over a field $k$ is a separated scheme of finite type over $k$. Unless otherwise stated, we shall assume varieties to be geometrically integral. For a smooth variety $X$ we write $\omega_X$ for the canonical sheaf of $X$, and $K_X$ for its class in $\text{Pic}X$.

Scheme: Locally ringed space that is locally a prime spectrum of a commutative ring.

Ringed Space: $(X, \mathcal{O}_X)$ is a topological space $X$ together with a sheaf of rings $\mathcal{O}_X$ on $X$. The sheaf $\mathcal{O}_X$ is called the structure sheaf of $X$.

Locally Ringed Space: $(X, \mathcal{O}_X)$ ringed space such that all stalks of $\mathcal{O}_X$ are local rings (i.e. they have unique maximal ideals).

Stalk: The stalk of a sheaf $\mathcal{F}$ at a point $x$, usually denoted by $\mathcal{F}_x$, is the direct limit $\mathcal{F}_x := \lim_{\overset{\longrightarrow}{U \ni x}} \mathcal{F}(U)$. An element of the stalk is an equivalence class of elements of $X_U \in \mathcal{F}(U)$, where two sections $x_U$ and $x_V$ are considered equivalent if the restriction of the two sections coincide on some neighborhood of $x$.

Sheaves: Presheaves: Given a topological set $X$ and a category $\mathcal{C}$, a presheaf on $X$ is a functor $\mathcal{F}$ with values in $\mathcal{C}$ given by the following data:

- $\forall U \subseteq X$ open set, there corresponds an object $\mathcal{F}(U)$ in $\mathcal{C}$;
- $\forall V \subseteq U$ inclusion of open sets, there corresponds a morphism $\text{res}_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ (restriction morphism).

Sheaves: A sheaf is a presheaf with values in the category of sets that satisfies the following two axioms:

- (Locality) If $(U_i)$ is an open covering of an open set $U$, and if $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ $\forall U_i$, then $s = t$;
- (Gluing) If $(U_i)$ is an open covering of an open set $U$, and if $\forall i$ a section $s_i \in \mathcal{F}(U_i)$ is given, such that for each pair $U_i, U_j$ of covering sets the restriction of $s_i$ and $s_j$ agree on the overlaps, i.e. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each $i$.

Prime Spectrum: Also called spectrum of a ring $R$, and denoted by $\text{Spec}(R)$, is the set of all prime ideals of $R$, equipped with the Zariski topology. This is defined by the closed sets being $V(I) = \{P \in \text{Spec}(R) \mid I \subseteq P\}$, where $I$ is any ideal in $R$.

Separated Scheme: A scheme $X$ over a ring $R$ is separated if the map $X \to X \times_R X$ is a closed immersion.

Scheme over a Ring: A scheme $X$ over a ring $R$ is a scheme $X$ endowed with a morphism $X \to \text{Spec}R$. 

Definition. A Variety $X$ over a field $k$ is a separated scheme of finite type over $k$. Unless otherwise stated, we shall assume varieties to be geometrically integral. For a smooth variety $X$ we write $\omega_X$ for the canonical sheaf of $X$, and $K_X$ for its class in $\text{Pic}X$. 

Scheme: Locally ringed space that is locally a prime spectrum of a commutative ring.

Ringed Space: $(X, \mathcal{O}_X)$ is a topological space $X$ together with a sheaf of rings $\mathcal{O}_X$ on $X$. The sheaf $\mathcal{O}_X$ is called the structure sheaf of $X$.

Locally Ringed Space: $(X, \mathcal{O}_X)$ ringed space such that all stalks of $\mathcal{O}_X$ are local rings (i.e. they have unique maximal ideals).

Stalk: The stalk of a sheaf $\mathcal{F}$ at a point $x$, usually denoted by $\mathcal{F}_x$, is the direct limit $\mathcal{F}_x := \lim_{\overset{\longrightarrow}{U \ni x}} \mathcal{F}(U)$. An element of the stalk is an equivalence class of elements of $X_U \in \mathcal{F}(U)$, where two sections $x_U$ and $x_V$ are considered equivalent if the restriction of the two sections coincide on some neighborhood of $x$.

Sheaves: Presheaves: Given a topological set $X$ and a category $\mathcal{C}$, a presheaf on $X$ is a functor $\mathcal{F}$ with values in $\mathcal{C}$ given by the following data:

- $\forall U \subseteq X$ open set, there corresponds an object $\mathcal{F}(U)$ in $\mathcal{C}$;
- $\forall V \subseteq U$ inclusion of open sets, there corresponds a morphism $\text{res}_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ (restriction morphism).

Sheaves: A sheaf is a presheaf with values in the category of sets that satisfies the following two axioms:

- (Locality) If $(U_i)$ is an open covering of an open set $U$, and if $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ $\forall U_i$, then $s = t$;
- (Gluing) If $(U_i)$ is an open covering of an open set $U$, and if $\forall i$ a section $s_i \in \mathcal{F}(U_i)$ is given, such that for each pair $U_i, U_j$ of covering sets the restriction of $s_i$ and $s_j$ agree on the overlaps, i.e. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each $i$.

Prime Spectrum: Also called spectrum of a ring $R$, and denoted by $\text{Spec}(R)$, is the set of all prime ideals of $R$, equipped with the Zariski topology. This is defined by the closed sets being $V(I) = \{P \in \text{Spec}(R) \mid I \subseteq P\}$, where $I$ is any ideal in $R$.

Separated Scheme: A scheme $X$ over a ring $R$ is separated if the map $X \to X \times_R X$ is a closed immersion.

Scheme over a Ring: A scheme $X$ over a ring $R$ is a scheme $X$ endowed with a morphism $X \to \text{Spec}R$. 


1.1 Algebraic K3 surfaces

**Scheme of Finite Type:** A scheme \( X \) over a ring \( R \) is of finite type over \( R \) if \( X \) admits a finite covering \( X = \bigcup U_i \) with \( U_i = \text{Spec} A_i \), such that the \( A_i \) are algebras of finite type (i.e. finitely generated as algebras) over \( R \).

**Geometrically Integral Scheme:** A scheme that is both reduced (the local rings are reduced rings, i.e. they don’t have nilpotent elements) and irreducible (is not the union of two proper subsets) is called integral. A scheme \( X \) over \( k \) is geometrically integrally (over \( k \)) if the scheme \( X_{k'} \) is integral for every field extension \( k' \) of \( k \).

**Canonical Sheaf:** The canonical bundle (or sheaf) of a smooth variety \( X \) of dimension \( n \) is the \( n \)-th exterior power of the cotangent bundle of \( X \).

**Pic(\( X \))** The Picard group \( \text{Pic}(X) \) of a variety \( X \) is the group of isomorphisms classes of line bundles (invertible sheaves) on \( X \), with the operation being the tensor product.

**Definition.** An algebraic K3 surface is a smooth projective 2-dimensional variety over a field \( k \) such that \( \omega_X \cong \mathcal{O}_X \) and \( H^1(X, \mathcal{O}_X) = 0 \).

**Example (K3 surfaces of degree 4, 6, and 8).** Let \( X \) be a smooth complete intersection of type \((d_1, \ldots, d_r)\) in \( \mathbb{P}^n_k \). Without loss of generality, we can assume \( d_i \geq 2 \) for every \( i \). Then \( X \) becomes a K3 surface only when

\[
\begin{align*}
  n &= 3, \quad d_1 = 4, \\
  n &= 4, \quad (d_1, d_2) = (2, 3), \\
  n &= 5, \quad (d_1, d_2, d_3) = (2, 2, 2)
\end{align*}
\]

and all these complete intersections are indeed K3 surfaces.

**Proof.** A complete intersection \( X \subseteq \mathbb{P}^n \) is a variety of dimension \( r \) such that \( X = H_1 \cap \cdots \cap H_r \) is the intersection of exactly \( r \) hypersurfaces. If one of the \( d_i = 1 \) is equal to one, then \( H_i \cong \mathbb{P}^{n-1} \) is isomorphic to a projective space of dimension \( n - 1 \), and hence \( X \) is in fact a subvariety of \( \mathbb{P}^{n-1} \) which is a complete intersection in \( \mathbb{P}^{n-1} \), intersection of \( r - 1 \) hypersurfaces. Hence we can assume \( d_i \geq 2 \) for every \( i \). In [Hartshorne, Ex. 8.4 (e)] it is stated that if \( X \) is a complete intersection in \( \mathbb{P}^n \), then

\[
\omega_X \cong \mathcal{O}_X \left( \sum d_i - n - 1 \right).
\]

The request \( \omega_X \cong \mathcal{O}_X \) and with the request of \( X \) being of dimension two force

\[
\begin{cases}
  r = n - 2 \\
  \sum d_i = n + 1
\end{cases}
\]

Using that \( d_i \geq 2 \) we only have finitely many possibilities, namely

\[
\begin{align*}
  n &= 3, \quad d_1 = 4, \\
  n &= 4, \quad (d_1, d_2) = (2, 3), \\
  n &= 5, \quad (d_1, d_2, d_3) = (2, 2, 2)
\end{align*}
\]

as claimed. Now we come to the proof of the fact that \( H^1(X, \mathcal{O}_X) = 0 \). To this end we invoke Lefschetz hyperplanes theorem, or better one of its corollaries, that is stated as an example in [16] (see [16 Example 3.1.24]).
Theorem. Let $Y$ be a smooth projective variety of dimension $n$, and $D$ a non-singular ample effective divisor on $X$. Denote by

$$r_{q,p} : H^q(Y, \Omega^p_Y) \rightarrow H^q(D, \Omega^p_D)$$

the natural maps determined by restriction. Then

$$r_{q,p} \text{ is } \begin{cases} \text{an isomorphism} & \text{for } p + q \leq n - 2, \\ \text{injective} & \text{for } p + q = n - 1. \end{cases}$$

In our situation we set $Y = \mathbb{P}^3$ and $Y = X$, which is a non-singular ample effective divisor. We have then $n = 3$ and we look at the pair $(p, q) = (0, 1)$ (which is allowed since $p + q = 1 = n - 2$). Observe that for $p = 0$ it is $\Omega^0_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3} = \mathcal{O}$ and $\Omega^0_X = \mathcal{O}_X$. Then by the theorem we have

$$H^1(\mathbb{P}^3, \mathcal{O}) \cong H^1(X, \mathcal{O}_X).$$

Since the term on the left is trivial, it follows that we have $H^1(X, \mathcal{O}_X) = 0$, and this concludes the proof of the exercise.

Example (degree 2 K3 surfaces). Suppose that $\text{char}(k) \neq 2$. Let $X$ be a double cover of $\mathbb{P}^2$ branched along a smooth sextic curve $C$ in $\mathbb{P}^2$. Then $X$ is a K3 surface.

Proof. For a sextic curve $C$ in $\mathbb{P}^2$, given by a homogeneous degree six polynomial $f(x_0, x_1, x_2)$, we can consider the set $X$ given by the points

$$X := \{ (x_0, x_1, x_2, w) : w^2 = f(x_0, x_1, x_2) \}.$$

The set $X$ lives naturally in the weighted projective space $\mathbb{P}(1, 1, 1, 3)$. If the curve $C$ is non-singular, then $X$ is non-singular. Moreover, to every point $p \in \mathbb{P}^2 \setminus C$ corresponds a pair of points in $X$ that project on $p$, while for each $q \in C$ there corresponds only one point in $X$. This shows that $X$ is a 2-dimensional variety which is a double cover of $\mathbb{P}^2$, ramified along the smooth sextic curve $C$. To show that $X$ is a K3 surface we need to check that $\omega_X \cong \mathcal{O}_X$ and that $H^1(X, \mathcal{O}_X) = 0$. We will use the following lemma, which we take from [2] (see [2, I, §17, Lemma (17.1)]).

Lemma. Let $X \rightarrow Y$ be a double covering of $Y$ branched along a smooth divisor $B$ and determined by $\mathcal{L}$, where $\mathcal{L} \otimes^2 = \mathcal{O}_Y(B)$. Then

$$K_X = \pi^*(K_Y \otimes \mathcal{L}).$$

In our case we have $Y = \mathbb{P}^2$ and $B = C$, so that $\mathcal{O}_{\mathbb{P}^2}(B) = \mathcal{O}_{\mathbb{P}^2}(6) = \mathcal{L} \otimes^2$ and consequently $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(3)$. This gives

$$\omega_X \cong \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{L}) \cong \pi^*(\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(3)) \cong \pi^*(\mathcal{O}_{\mathbb{P}^2}) = \mathcal{O}_X.$$

Now we need to show that $H^1(X, \mathcal{O}_X) = 0$. We use [10, III, §8, Ex. 8.2], which we state below.
1.2 Complex K3 surfaces

Exercise. Let \( f : X \rightarrow Y \) be an affine morphism of schemes with \( X \) noetherian, and let \( \mathcal{F} \) be a quasi-coherent sheaf on \( X \). Then \( H^i(X, \mathcal{F}) \cong H^i(Y, f_* \mathcal{F}) \) for every \( i \geq 0 \).

In our case it is \( Y = \mathbb{P}^2 \) and \( X \) is noetherian, which gives
\[
H^1(X, \mathcal{O}_X) \cong H^1(\mathbb{P}^2, \mathcal{O}) = 0.
\]
This concludes the proof of the fact that \( X \) is a K3 surface.

Example (Kummer surfaces). Let \( k \) be a field of characteristic different from 2. Let \( A \) be an abelian surface over \( k \). Consider the involution \( \iota : A \rightarrow A \) given by \( \iota : x \mapsto -x \). The fixed part of this involution consists of 16 \( \bar{k} \)-points. Let \( \rho : \tilde{A} \rightarrow A \) be the blow up of this \( k \)-scheme. The involution \( \iota \) lifts to the involution \( \tilde{i} \) on \( \tilde{A} \), and we consider the quotient \( \pi : \tilde{A} \rightarrow \tilde{A}/i =: X \). Then \( X \) is a K3 surface and it is called a Kummer surface.

Proye. We recall another result from Hartshorne’s book (see \([10, V, \S 3, \text{Proposition 3.3}] \)), that expresses the canonical sheaf of a blow up of a variety at a point in terms of the canonical sheaf of the original variety and the exceptional divisors.

Proposition. Let \( X \) be a surface, \( P \) a point in \( X \), and \( \tilde{X} \rightarrow X \) the blow up of \( X \) at \( P \). Denote by \( E \) the exceptional divisor above \( P \). Then the canonical divisor of \( \tilde{X} \) is given by \( K_{\tilde{X}} = \pi^* K_X + E \).

I want to conclude that
\[
\omega_{\tilde{A}} \cong \mathcal{O}(\sum E_i) \quad \text{and} \quad \omega_{\tilde{A}} \cong \pi^* \omega_X \otimes \mathcal{O}(\sum E_i)
\]
and from there deduce, following Huybrechts, that \( \omega_X \cong \mathcal{O}_X \). For the cohomology group \( H^1(X, \mathcal{O}_X) \) I can’t do else than quoting Huybrechts. So \( \omega_X \cong \mathcal{O}_X \) and \( H^1(X, \mathcal{O}_X) = 0 \) and this proves that \( X \) is a K3 surface.

1.2 Complex K3 surfaces

In this section, we assume that our ground field is the field of complex numbers \( \mathbb{C} \). We study a notion of K3 surfaces for complex manifolds.

Definition A complex K3 surface is a compact connected 2-dimensional complex manifold \( X \) such that \( \omega_X \cong \mathcal{O}_X \) and \( H^1(X, \mathcal{O}_X) = 0 \).

For any separated scheme of finite type over \( \mathbb{C} \), one can associate a complex analytic space \( X^{an} \) whose underlying space is \( X(\mathbb{C}) \) with the classical topology. Moreover, for any coherent sheaf \( \mathcal{F} \) on \( X \), one can associate the analytic coherent sheaf \( \mathcal{F}^{an} \). When \( X \) is a projective variety, this functor \( \mathcal{F} \mapsto \mathcal{F}^{an} \) gives an equivalence
between the category of coherent sheaves on $X$, and the category of analytic coherent sheaves on $X^{\text{an}}$. (GAGA principle) Furthermore, this gives us an isomorphism of cohomologies

$$H^q(X, \mathcal{F}) \rightarrow H^q(X^{\text{an}}, \mathcal{F}^{\text{an}}).$$

This discussion implies the following result:

**Proposition 1.9.** Let $X$ be an algebraic K3 surface over $\mathbb{C}$. Then $X^{\text{an}}$ is a complex K3 surface.

**Example** There is a complex K3 surface which is not projective. (Actually for smooth complete surfaces, algebraicity and projectivity are equivalent.) Let $T$ be a 2-dimensional complex torus over $\mathbb{C}$. One can apply the construction of Kummer surfaces to $T$, and one obtains a complex K3 surface $X$. Then $X$ is projective if and only if $T$ is projective. Most of 2-dimensional complex tori are not projective.

GAGA principle also tells us that a complete analytic space is a projective variety if and only if it admits an embedding into a projective space. Thus the above functor gives us the full proper embedding of the category of algebraic K3 surfaces into the category of complex K3 surfaces.

# Lecture 2: Topology of K3 surfaces: Annelies Jaspers (06/10)

We mainly follow Section 1 of [30]. Also see [13]. In this section, $X$ denotes a complex K3 surface.

## 2.1 Singular cohomologies of complex K3 surfaces

We denote the topological Euler characteristic of a space by $e(\cdot)$, and $c_i(X)$ denotes the $i$-th Chern class of the tangent bundle of $X$.

**Exercise 2.1.** Use Noether’s formula to conclude that $e(X) = c_2(X) = 24$.

**Exercise 2.2.** Show that $H^0(X, \mathbb{Z}) = H^4(X, \mathbb{Z}) = \mathbb{Z}$.

**Exercise 2.3.** Use the exponential sequence to prove that $H^1(X, \mathbb{Z}) = 0$.

**Exercise 2.4.** Use Poincaré duality to prove that $H^3(X, \mathbb{Z})$ is a torsion group and $H^3(X, \mathbb{Z})_{\text{tor}} \cong H_1(X, \mathbb{Z})_{\text{tor}}$.

**Exercise 2.5.** Use the universal coefficients theorem to prove that $H_1(X, \mathbb{Z})_{\text{tor}}$ is dual to $H^2(X, \mathbb{Z})_{\text{tor}}$. 
2.2 Simply connectedness

Exercise 2.6. Show that \( H_1(X, \mathbb{Z})_{\text{tor}} = 0 \). (See Proposition 1.11 of [30]).

All together we conclude that any singular cohomology of a complex K3 surface is a free abelian group. Moreover, since \( e(X) = 24 \), the rank of \( H^2(X, \mathbb{Z}) \) is \( 24 - 1 - 1 = 22 \). The cup product induces a pairing \( B \) on \( H^2(X, \mathbb{Z}) \).

Proposition 2.7. The pairing \( B \) is even, unimodular, and has signature \( (3, 19) \).

Proof. This follows from Wu’s formula, Poincaré duality, and the Thom-Hirzebruch index theorem. See [30, p. 5], [2, p.310], or [13, Proposition 3.5]. □

Corollary 2.8. Let \( X \) be a complex K3 surface. Then there is an isometry

\[
H^2(X, \mathbb{Z}) \cong U \oplus U \oplus E_8(-1) \oplus E_8(-1),
\]

where \( U \) is the hyperbolic plane and \( E_8(-1) \) is the unique even unimodular negative definite lattice of rank 8.

Proof. See [30, Theorem 1.13]. □

2.2 Simply connectedness

The exponential sequence gives rise an exact sequence:

\[
0 \to \text{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X).
\]

The second map can be identified with the first Chern form \( c_1(L) \). This exact sequence shows that we have an isomorphism

\[
\text{Pic}(X) \cong \text{NS}(X).
\]

However \( \text{Pic}(X) \) could differ from the group of numerical classes.

Theorem 2.9. Every complex K3 surface is simply connected.

Proof. This follows from the fact that all complex K3 surfaces are diffeomorphic to each other. The idea is the following:

- every complex Kummer surface is diffeomorphic (easy to see)
- there is an open set in the period domain around the period point of a K3 surface where the K3 surface can be deformed,
- projective Kummer surfaces are dense in the period domain.

We will postpone the verification of these facts to later sections. Now our assertion follows from the following proposition: □

Proposition 2.10. Any smooth quartic surface is simply connected.

Proof. This follows from Lefschetz hyperplane theorem.
2.3 Appendix by Annelies Jaspers and Nadim Rustom

2.3.1 Complex K3 surfaces

There are two classes of objects, algebraic and analytic, both called K3 surfaces.

**Definition** An algebraic K3 surface is a smooth, projective 2-dimensional variety over a field $k$ such that

$$\omega_X \cong \mathcal{O}_X$$

and

$$H^1(X, \mathcal{O}_X) = 0$$

where $\omega_X$ is the canonical sheaf.

**Definition** A complex K3 surface is a compact, connected, complex manifold of dimension 2 such that

$$\Omega^2_X \cong \mathcal{O}_X$$

and $H^1(X, \mathcal{O}_X) = 0$, where $\Omega_X$ is the sheaf of holomorphic 1-forms, and $\mathcal{O}_X$ is the sheaf of holomorphic functions on $X$.

These definitions are related through the GAGA principle.

2.3.2 Serre’s GAGA principle

Let $X$ be a scheme of finite type over $\mathbb{C}$. To $X$ we associate a complex analytic space $X^{an}$ whose underlying topological space is $X(\mathbb{C})$ with the complex topology, and a continuous map

$$X^{an} \rightarrow \phi X$$

of locally ringed spaces. The association $X \mapsto X^{an}$ is a functor, and it is well behaved geometrically:

- $X$ connected $\iff X^{an}$ connected.
- $X$ smooth $\iff X^{an}$ complex manifold.
- $X$ has dimension $n$ $\iff X^{an}$ has dimension $n$.
- $X$ separated $\iff X^{an}$ Hausdorff.
- $X$ proper $\iff X^{an}$ compact.

meaning locally it is diffeomorphic to $\mathbb{C}^2$. 
Recall that if a \((Y, \mathcal{O}_Y)\) is a locally ringed space, then a sheaf \(\mathcal{F}\) of \(\mathcal{O}_Y\)-modules on \(Y\) is said to be coherent if there exists integers \(p, q \geq 0\) such that locally we have a short exact sequence:

\[
\mathcal{O}_Y^p \to \mathcal{O}_Y^q \to \mathcal{F} \to 0.
\]

We then denote by \(\text{Coh}_Y\) the category of coherent sheaves on \(Y\).

The map \(\varphi\) induces a functor

\[
\text{Coh}_X \to \text{Coh}_{\text{an}} X, \quad \mathcal{F} \mapsto \varphi^* \mathcal{F} =: \mathcal{F}^\text{an}.
\]

We have \(\mathcal{O}_{\text{an}} X = \mathcal{O}_{X} \text{an}\) and \(\Omega_{\text{an}} X = \Omega_{X} \text{an}\).

**Theorem 2.11.** If \(X\) is projective, then the functor \(\text{an}\)– is an equivalence of categories. Moreover,

\[
H^i(X, \mathcal{F}) \cong H^i(\text{an} X, \mathcal{F}^\text{an}).
\]

So we have:

**Theorem 2.12.** Therefore, if \(X\) is an algebraic K3 surface over \(\mathbb{C}\), then \(X_{\text{an}}\) is a complex K3 surface.

**Remark** Not every complex K3 surface is the analytification of an algebraic K3 surface. For example, if \(T\) is a 2-dimensional complex torus, by Kummer’s construction we obtain a complex K3 surface \(X\). Then \(X\) is projective if and only if \(T\) is projective. But there are 2-dimensional complex tori which are not projective.

### 2.3.3 Singular cohomology

**Theorem 2.13.** Let \(X\) be a complex K3 surface. Then

- \(H^0(X, \mathbb{Z}) \cong \mathbb{Z}\).
- \(H^1(X, \mathbb{Z}) \cong 0\).
- \(H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}\).
- \(H^3(X, \mathbb{Z}) \cong 0\).
- \(H^4(X, \mathbb{Z}) \cong \mathbb{Z}\).

Note that this would mean that the Euler characteristic

\[
e(X) = \sum_{i=0}^{4} (-1)^i \text{rank} H^i(X, \mathbb{Z}) = 24.
\]

We can calculate the Euler characteristic separately using Noether’s formula:
\[ \chi(X, \mathcal{O}_X) = \frac{1}{12} \left( c_1(X)^2 + c_2(X) \right) \]

where \( c_i(X) \) is the \( i \)th Chern class of \( X \). Note that \( c_1(X)^2 = K_X^2 \), i.e. the self-intersection number of the canonical divisor \( K_X \), and \( c_2(X) = e(X) \). Therefore

\[ \chi(X, \mathcal{O}_X) = \frac{e(X)}{12}. \]

But

\[ \chi(X, \mathcal{O}_X) = \sum_{i=0}^{2} (-1)^i \dim H^i(X, \mathcal{O}_X) \]

\[ = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) + \dim H^2(X, \mathcal{O}_X) \]

\[ = 2 \dim H^0(X, \mathcal{O}_X) = 2 \]

since \( H^1(X, \mathcal{O}_X) = 0 \), and \( \dim H^2(X, \mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) \) by Serre duality, and \( \dim H^0(X, \mathcal{O}_X) = 1 \) because \( X \) is projective.

\( H^0(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z} \) This is because \( X \) is connected and orientable.

\( H^1(X, \mathbb{Z}) \) We use the short exact sequence of sheaves

\[ 0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{f \mapsto e^{2\pi i f}} \mathcal{O}_X^* \to 0 \]

where \( \mathbb{Z} \) is the locally constant sheaf with values in \( \mathbb{Z} \). Taking the long exact sequence in cohomology we get

\[ H^0(X, \mathcal{O}_X) \xrightarrow{\alpha} H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X). \]

But \( H^0(X, \mathcal{O}_X) = \mathbb{C}, \ H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*, \) and the map \( \alpha \) is the exponential map, hence \( \alpha \) is surjective. Since \( H^1(X, \mathcal{O}_X) = 0 \), we conclude that

\[ H^1(X, \mathbb{Z}) = H^1(X, \mathbb{Z}) = 0. \]

\( H^3(X, \mathbb{Z}) \) By Poincare duality,

\[ H^3(X, \mathbb{Z}) \cong H_1(X, \mathbb{Z}). \]

We know that \( \text{rank} H_1(X, \mathbb{Z}) = \text{rank} H^1(X, \mathbb{Z}) = 0 \), hence \( H^3(X, \mathbb{Z}) \) is torsion.

**Proposition 2.14.** \( H_1(X, \mathbb{Z}) = 0. \)
Proof. Let $\alpha \in H_1(X, \mathbb{Z})$. Then $\alpha$ is torsion. Let $n$ be the order of $\alpha$. This gives a surjection

$$H_1(X, \mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}.$$ 

Recall that $H_1(X, \mathbb{Z})$ is the abelianisation of the fundamental group $\pi_1(X, x)$ (for any choice of basepoint $x \in X$), so we have a composite map:

$$\pi_1(X, x) \to H_1(X, \mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}.$$ 

The kernel of this composite is a subgroup of $\pi_1(X, x)$ of index $n$. The Galois correspondence tells us that there is a 1-1 correspondence between (conjugacy classes of) subgroups of $\pi_1(X, x)$ of order $n$ and (isomorphism classes of) unramified coverings of $X$ of degree $n$. We will show that $X$ admits no non-trivial unramified coverings.

Let $\pi : Y \to X$ be an unramified covering of degree $n$. Then comparing Euler characteristics,

$$e(Y) = ne(X) = 24n$$ 

Hurwitz’s formula gives $\omega_Y = \pi^*\omega_X \cong \mathcal{O}_Y$. Noether’s formula gives

$$\chi(Y, \mathcal{O}_Y) = \frac{1}{12}(c_1(Y)^2 + c_2(Y)) = 2n.$$ 

On the other hand,

$$\chi(Y, \mathcal{O}_Y) = h^0(Y, \mathcal{O}_Y) - h^1(Y, \mathcal{O}_Y) + h^2(Y, \mathcal{O}_Y)$$

$$= 2 - h^1(Y, \mathcal{O}_Y)$$ 

since by Serre duality, $h^2(Y, \mathcal{O}_Y) = h^0(Y, \mathcal{O}_Y)$. We find that

$$h^1(Y, \mathcal{O}_Y) = 2 - 2n \geq 0$$ 

and therefore $n = 1$, allowing us to conclude that $\alpha = 0$. 

$H^2(X, \mathbb{Z})$ From the calculation of the Euler characteristic of $X$, we know that $\text{rank} H^2(X, \mathbb{Z}) = 22$. We need to show that $H^2(X, \mathbb{Z})$ is free. From the Universal Coefficient Theorem, we have a short exact sequence

$$0 \to \text{Ext}^1(H_1(X, \mathbb{Z}), \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to \text{Hom} H_2(X, \mathbb{Z}) \mathbb{Z} \to 0.$$ 

Since we have shown $H_1(X, \mathbb{Z}) = 0$, we conclude

$$H^2(X, \mathbb{Z}) \cong \text{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) \cong H^2(X, \mathbb{Z})_{\text{free}}$$ 

and hence $H^2(X, \mathbb{Z})$ is free.
2.3.4 Cup product

From topology we have the cup-product

\[ B_{\text{top}} : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \cong \mathbb{Z}. \]

But we can identify \( H^2(X, \mathbb{Z}) \) with a subspace of \( H^2(X, \mathbb{R}) \) which is the De Rham cohomology group, and which carries the cup product:

\[ (\alpha, \beta) \in H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \mapsto B_{\text{dR}}(\alpha, \beta) = \int_X \alpha \wedge \beta. \]

Moreover, the identification \( H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R}) \) is compatible with the respective cup products. Thus \( B_{\text{dR}} \) restricts on \( H^2(X, \mathbb{Z}) \) to \( B = B_{\text{top}} \).

**Proposition 2.15.** The pairing \( B \) is perfect, even \( B(x, x) \in 2\mathbb{Z} \), uni-modular, of signature \((3, 19)\).

**Proof.**
- Even: Wu’s formula.
- Uni-modular: Poincaré duality.
- Signature \((3, 19)\): Thom-Hirzebruch index formula: let \( b_+ \) be the number of positive eigenvalues, \( b_- \) the number of negative eigenvalues. Then \( b_+ + b_- = 22 \). Thom-Hirzebruch formula gives

\[ b_+ - b_- = \frac{1}{3}(c_1(X)^2 - 2c_2(X)) = -16. \]

\[ \Box \]

**Corollary 2.16.** \((H^2(X, \mathbb{Z}), B)\) is isometric to the K3 lattice

\[ \Lambda_{K3} := U \oplus U \oplus U \oplus E_8(-1) \oplus E_28(-1) \]

where \( U \) is the hyperbolic plane \( \mathbb{Z}^2 \) and \( E_8(-1) \) is the unique even unimodular negative definite lattice of rank 8.

2.3.5 Simply connectedness

**Theorem 2.17.** All complex K3 surfaces are simply connected.

**Proof.** The proof is in two parts
- All complex K3 surfaces are diffeomorphic (!!).
- Let \( X \) be a smooth quartic in \( \mathbb{P}^3_{\mathbb{C}} \). Let \( v \) be the Veronese 4-uple embedding

\[ v : \mathbb{P}^3_{\mathbb{C}} \to \mathbb{P}^{34}_{\mathbb{C}}. \]

Then \( v(X) \) is a hyperplane in \( v(\mathbb{P}^3_{\mathbb{C}}) \). Now the Lefschetz hyperplane theorem gives

\[ \pi_1(X, -) = \pi_1(v(X), -) = \pi_1(v(\mathbb{P}^3_{\mathbb{C}})) = 0. \]

\[ \Box \]
3 Lecture 3: Hodge structures: Simon Rose (20/10)

Here we follow [13, Chapter 3]. Also see [30, Section 1]. We assume that our ground field is \( \mathbb{C} \).

3.1 Hodge structures

**Definition** Let \( V \) be a free abelian group of finite rank or a finite dimensional vector space over \( \mathbb{Q} \). A Hodge structure of weight \( n \in \mathbb{Z} \) on \( V \) is given by a direct sum decomposition of the complex vector space \( V_{\mathbb{C}} \):

\[
V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}
\]

such that \( V^{p,q} = V^{q,p} \).

For any integral (or rational) Hodge structure \( V \) of even weight \( n = 2k \), the intersection \( V \cap V^{k,k} \) is called the space of Hodge classes.

**Example** The Tate Hodge structure is denoted by \( \mathbb{Z}(1) \). It is the Hodge structure of weight \(-2\) on the free \( \mathbb{Z} \)-module of rank one \((2\pi i)\mathbb{Z}\).

**Definition** Any compact complex manifold admits a hermitian metric and any such metric can be given by its associated \((1,1)\)-form \( \alpha = \sum \alpha_{i,j} dz_i \wedge d\bar{z}_j \). A real \((1,1)\)-form is positive if the associated hermitian form \( \alpha_{i,j} \) is positive definite. The \((1,1)\)-form associated to a hermitian metric is positive and vice versa. A metric whose \((1,1)\)-form is closed is called a Kähler metric. A compact complex manifold with a Kähler metric is called a Kähler manifold.

**Exercise 3.1.** Show that any projective manifold is a Kähler manifold. (Hint: Any submanifold of a Kähler manifold is again Kähler. The projective space \( \mathbb{P}^n \) is Kähler since it comes with the Fubini-study metric which is Kähler.)

**Exercise 3.2.** Show that any complex torus is a Kähler manifold.

This example shows that in general, a Kähler manifold may not be projective. However, we have the following criterion:

**Theorem 3.3.** Let \( X \) be a Kähler manifold. Suppose that the cohomology class of a Kähler metric is integral. Then \( X \) is projective.

The following theorem is super important for us.

**Theorem 3.4.** (Siu) Every complex K3 surface is Kähler.

*Proof. See [24] or [2, IV-3].*
Theorem 3.5. (Hodge decomposition) For a compact Kähler manifold \( X \), \( H^n(X, \mathbb{Z}) \) admits a natural Hodge decomposition of weight \( n \) given by
\[
H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)
\]
where \( H^{p,q}(X) \) could either be viewed as the space of de Rham classes of bidegree \((p,q)\) or as the Dolbeault cohomology \( H^q(X, \Omega^p_X) \).

Exercise 3.6. Prove this statement for any complex torus.

Exercise 3.7. Describe the Hodge diamond for complex K3 surfaces.

Let \( X \) be a smooth projective variety. For any subvariety \( Z \subset X \) of codimension \( k \), one can define the fundamental class \([Z] \in H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X)\).

Conjecture 3.8. (Hodge conjecture) For a smooth projective variety \( X \), the subspace of \( H^{2k}(X, \mathbb{Q}) \) spanned by all algebraic cycles coincides with the space of Hodge classes.

This conjecture is true for \( k = 1 \) and \( d - 1 \) where \( d = \dim X \).

For any rational Hodge structure, the Weil operator \( C \) is a \( \mathbb{R} \)-linear map on \( V_\mathbb{R} \) which acts on \( V^{p,q} \) by multiplication with \( i^{p-q} \). It clearly preserves the real vector space \((V^{p,q} \oplus V^{q,p}) \cap V_\mathbb{R}\).

Definition A polarization of a rational Hodge structure \( V \) of weight \( n \) is a morphism of Hodge structures
\[
\psi : V \otimes V \to \mathbb{Q}(-n)
\]
such that its \( \mathbb{R} \)-linear extension yields a positive definite symmetric form
\[
(v, w) \mapsto \psi(v, Cw)
\]
on \((V^{p,q} \oplus V^{q,p}) \cap V_\mathbb{R}\). The pair \((V, \psi)\) is called a polarized Hodge structure. An isomorphism of Hodge structures which is compatible with polarizations is called a Hodge isometry.

3.2 Hodge structures of weight one

Example Let \( X \) be a smooth projective variety. For any rational Kähler class \( \omega \in H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \), the alternation pairing
\[
\psi(v, w) = \int_X v \wedge w \wedge \omega^{d-1}
\]
is a polarization on \( H^1(X, \mathbb{Q}) \).
**Exercise 3.9.** Show that there is one to one correspondence between the set of isomorphism classes of integral Hodge structures of weight one and the set of isomorphism classes of complex tori.

Moreover, we have the following theorem:

**Theorem 3.10.** The above correspondence gives us one to one correspondence between the set of isomorphism classes of abelian varieties and the set of isomorphism classes of polarizable integral Hodge structures of weight one.

**Theorem 3.11.** (Global Torelli theorem for curves) Two smooth compact complex curves are isomorphic if and only if their polarized Hodge structures are isometric.

### 3.3 Hodge structures of weight two

**Definition** A Hodge structure $V$ of weight two is of K3 type if

$$\dim_\mathbb{C} V^{2,0} = 1 \text{ and } V^{p,q} = 0 \text{ for } |p-q| > 2.$$

**Example** For any complex K3 surface $X$, $H^2(X, \mathbb{Z})$ is a Hodge structure of K3 type.

**Example** For any hyperkähler manifold $X$ (a higher dimensional analogue of K3 surfaces), $H^2(X, \mathbb{Z})$ is a Hodge structure of K3 type.

**Example** For any smooth cubic fourfold $X$ in $\mathbb{P}^5$, $H^4(X, \mathbb{Z})$ admits a Hodge structure of K3 type.

For any complex K3 surface $X$, the transcendental lattice $T(X)$ is the orthogonal complement of $\text{NS}(X)$ in $H^2(X, \mathbb{Z})$ which admits a natural Hodge structure of K3 type.

Let $X$ be a complex K3 surface. Then the space $H^{2,0}(X)$ is generated by $\omega_X$. This satisfies the Hodge-Riemann relations:

1. $(\omega_X, \omega_X) = 0$;
2. $(\omega_X, \omega_X) > 0$;
3. $H^{2,0}(X) \oplus H^{0,2}(X)$ is orthogonal to $H^{1,1}(X)$.

Conversely if we have such $\omega_X$, then one can recover the Hodge structure of $X$. 
3.4 Appendix by Simon Rose

3.4.1 Hodge Structures

We begin by providing the definition of an integral Hodge structure. Note that there are corresponding definitions for rational or real Hodge structures.

**Definition** Let $V$ be a finitely generated free abelian group. Then a Hodge Structure of weight $k \in \mathbb{Z}$ on $V$ is a direct sum decomposition of $V_C := V \otimes \mathbb{C}$ into

$$V_C = \bigoplus_{p+q=k} V^{p,q}$$

where $V^{p,q} = V^{q,p}$.

**Remark** Note that we make no assumptions in this definition that $k, p, q \geq 0$. While most of our examples while have this restriction, it is not a necessary one. We should further note that since $V$ is assumed to be finitely generated, we will have $V^{p,q} = 0$ for $p, q \ll 0$ and $p, q \gg 0$.

We will furthermore throughout always assume that free abelian groups are finitely generated.

**Example** A simple, explicit example, is the following. Let $V$ be a free abelian group, and let $I : V \to V$ be an almost-complex structure. That is, $I$ is an endomorphism which satisfies $I^2 = -Id_V$. Then $V_C$ has a natural Hodge structure of weight 1 given by

$$V^{0,1} = \ker(I_C - \sqrt{-1}Id_V)$$

$$V^{1,0} = \ker(I_C + \sqrt{-1}Id_V)$$

i.e. the decomposition is given by the $\pm \sqrt{-1}$-eigenspaces of the complexified endomorphism $I_C$. These are clearly conjugate to each other, and so we have a Hodge structure as claimed.

**Example** A somewhat more exotic example is the following. The Tate-Hodge structure $\mathbb{Z}(1)$ is the unique Hodge structure of weight $-2$ on the free $\mathbb{Z}$-module $(2\pi \sqrt{-1})\mathbb{Z}$. This is useful in certain constructions.

For any integral or rational Hodge structure of even weight, the intersection $V \cap V^{k,k}$ is the space of Hodge classes. When we next discuss how the cohomology of certain complex manifolds yields naturally a collection of Hodge structures, we will find that the classes that correspond to submanifolds are all Hodge classes.

Let us describe this main source of Hodge structures.
**Definition** A complex manifold always admits a Hermitian metric. There is an associated (1,1)-form (the fundamental form) which can be written as

\[ \omega = \sum_{i,j} \omega_{i,j} \, dz_i \wedge d\bar{z}_j. \]

If this form is closed (i.e. \( d\omega = 0 \)), then we say that the metric is a Kähler metric, and we say that a manifold is Kähler if it admits a Kähler metric.

**Remark** The condition that \( d\omega = 0 \) is equivalent to the condition that the (1,1)-form is expressible in local coordinates as

\[ \omega = \frac{\sqrt{-1}}{2} \sum_{i,j} h^{i,j} \, dz_i \wedge d\bar{z}_j \]

with \( (h^{i,j}(z, \bar{z})) = 1 + O(|z|^2) \) is a positive-definite hermitian matrix.

Having introduced this definition, we should naturally ask for some examples of Kähler manifolds. Fortunately, such objects are manifold\(^\text{2}\). We begin with the following explicit example.

**Proposition 3.12.** Projective space \( \mathbb{P}^N \) is a Kähler manifold.

**Proof.** In this case, we can write down an explicit Kähler form, which comes from the Fubini-Study metric. If the coordinates on \( \mathbb{P}^N \) are given by \([z_0 : \cdots : z_N]\), and we define the standard charts to be \( U_i = \{ [z_0 : \cdots : z_N] \mid z_i \neq 0 \} \) with coordinates \( z_j/z_i \), then on \( U_i \) we can write the fundamental form as

\[ \omega_i = \frac{\sqrt{-1}}{2\pi} \frac{\partial \bar{\partial}}{\partial \bar{\partial} \log \left( \sum_{\ell=0}^N \frac{|z_\ell|^2}{|z_i|^2} \right)}. \]

One can check that this agrees on overlaps which then defines a form \( \omega \) on all of \( \mathbb{P}^N \), and moreover that this form satisfies \( d\omega = 0 \). \( \square \)

**Example** On \( \mathbb{P}^1 \), this form is given on the two charts as

\[ \omega_0 = \frac{\sqrt{-1}}{2\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} \]

and

\[ \omega_1 = \frac{\sqrt{-1}}{2\pi} \frac{dw \wedge d\bar{w}}{(1 + |w|^2)^2} \]

where \( w = z^{-1} \). We see obviously that in local coordinates, the form is of the form \( 1 + O(|z|^2) \) as required.

\(^2\)I’m sorry for this pun.
We now have some Kähler manifolds, but we now see that there are many, many more.

**Theorem 3.13.** Any submanifold of a Kähler manifold is also Kähler.

**Proof.** A complex submanifold inherits its metric and almost-complex structure from the ambient manifold. Furthermore, we have that

\[ dt^* \omega = i^* d\omega = 0 \]

and so the resulting manifold is also Kähler. \( \square \)

We find from this that every smooth projective variety—in particular, projective K3 surfaces—is Kähler. However, these are not the only manifolds which are Kähler.

**Proposition 3.14.** Let \( X = \mathbb{C}^N / \Lambda \) be a complex torus (i.e. \( \Lambda \) is a non-degenerate rank \( 2N \) sublattice of \( \mathbb{C}^N \)). Then \( X \) is Kähler.

**Proof.** Once again, we simply write down the form explicitly. In this case, we can write the fundamental form as

\[ \omega = \sum_{i=1}^{N} dz_i \wedge d\bar{z}_i. \]

We should note first of all that this is really a form on \( \mathbb{C}^N \); however, since it is translation-invariant, it descends to a form on the quotient. Second of all, it clearly satisfied \( d\omega = 0 \), and so the resulting quotient is Kähler. \( \square \)

One of the reasons this is of interest is that not all complex tori are projective; in fact, the generic complex torus of dimension greater than one will not be algebraic. From the above though, it is always Kähler.

These tori have a property that will be in some sense motivational for what follows.

**Theorem 3.15.** Let \( X = \mathbb{C}^n / \Lambda \) be a complex torus. Then \( H^k(X, \mathbb{Z}) \) carries a natural Hodge structure of weight \( k \) given by

\[ H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(\mathbb{C}^N / \Lambda) \]

where \( H^{p,q}(\mathbb{C}^N / \Lambda) \) is the space of \( (p, q) \)-forms; that is,

\[ H^{p,q}(\mathbb{C}^N / \Lambda) = \left\{ \sum_{i_1, \ldots, i_p} \lambda_{i_1, \ldots, i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \right\}. \]
The proof of this is essentially the same as the proof that tori are Kähler; the given forms are $\Lambda$-invariant and so descend to the quotient. Moreover, it is clear from the definition that $H^{p,q} = H^{q,p}$.

It turns out that for general Kähler manifolds, we have the same decomposition.

**Theorem 3.16.** Let $X$ be a Kähler manifold. Then $H^k(X, \mathbb{Z})$ carries a natural Hodge structure of weight $k$ given by

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X) = H^q(X, \Omega^p_X)$.

**Remark** One simple consequence that we obtain from this decomposition. Let $X$ be any Kähler manifold. Since we have a decomposition

$$H^1(X, \mathbb{C}) = H^{0,1}(X) \oplus H^{1,0}(X)$$

with $H^{0,1}(X) = H^{1,0}(X)$, it follows that they are vector spaces of the same dimension. In particular, we see that $b_1(X)$ must be even.

Remarkably, in the case of surfaces, this is a sufficient condition.

**Theorem 3.17.** Any smooth complex surface is Kähler if and only if $b_1(X)$ is even.

In particular, we knew from the above that the projective K3 surfaces were all Kähler; from this we can now conclude that since $b_1(X) = 0$, every K3 surface is Kähler.

### 3.4.2 The Hodge Diamond

One of the most basic invariants that we can consider for a manifold is its Betti numbers; that is, the dimension $b_k(X) = \dim_{\mathbb{R}} H^k(X, \mathbb{R}) = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$. When we have a Hodge decomposition, then we can refine this number.

**Definition** Let $X$ be a Kähler manifold, and define

$$h^k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$$

$$h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X).$$

From the Hodge decomposition, we find immediately that

$$h^k(X) = \sum_{p+q=k} h^{p,q}(X)$$
and that
\[
\chi(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X)
\]

We will now arrange these numbers in the **Hodge Diamond**. This is
\[
\begin{array}{ccccccc}
& & h^{n,n} & & & & \\
& h^{n-1,n} & & h^{n,n-1} & & & \\
. & & . & & & & . \\
. & & . & & & & . \\
h^{0,n} & h^{1,n-1} & \ldots & h^{n,0} & & & \\
. & & . & & & & . \\
h^{0,0} & & h^{1,0} & & & & h^{2,0}
\end{array}
\]

**Remark** There is a slight confusion in the literature of the exact orientation of this diamond: should $h^{0,0}(X)$ be at the top or the bottom? One would think that this is a pretty unambiguous thing, but as we will see the diagram has many symmetries and in particular, we have that $h^{n,n}(X) = h^{0,0}(X)$. As most authors simply provide the diagram in its most reduced and irredudant form, you can’t tell what the author actually means. Perhaps this is a reasonable way to avoid the deep schisms that could otherwise arise.

As stated just above, this diagram has many symmetries, the main three of which are the following.

**Proposition 3.18.** We have the following equalities of Hodge numbers.

1. $h^{p,q} = h^{q,p}$ due to complex conjugation. This says that the diagram is symmetric upon a reflection across the vertical axis.

2. $h^{p,q} = h^{n-p,n-q}$ which is due to Serre duality. That is, we have

\[
H^q(X, \Omega_X^p) \cong H^{n-q}(X, \Omega_X^{n-p})^\vee
\]

and this represents a rotation by an angle of $\pi$ through the center.
3. \( h^{p,q} = h^{n-q,n-p} \), which comes from the Lefschetz operator (cup product with power of a Kähler class). This says that the diagram is symmetric upon a reflection across a horizontal line through the middle.

That is, we have

Of course, any two of the described symmetries will generate the third.

Let us now look at some specific examples of Hodge diamonds.

**One-dimensional complex manifolds** Let \( C \) be a smooth compact curve of genus \( g \). All curves are projective, and so in particular are Kähler. Their Hodge decomposition is well known; in fact one can define the genus of the curve as

\[
g = h^0(C, \Omega_C) = h^{1,0}(C)
\]

This then yields that the Hodge diamond of a curve is given by

\[
\begin{array}{ccc}
1 & g & g \\
& & \\
1 & \\
\end{array}
\]

One should of course remark that while all curves of a fixed genus have the same Hodge diamond, they do not have the same Hodge decomposition. The position of \( H^{0,1}(C) \) within \( H^1(X, \mathbb{C}) \) varies from curve to curve, and in fact determines the complex structure (up to a change of integral basis).
Two-dimensional “Calabi-Yau” manifolds

In general, an \( n \)-dimensional Calabi-Yau manifold is a smooth complex manifold such that the canonical sheaf is trivial; that is, \( \Omega^n(X) \cong \mathcal{O}_X \). One may also impose extra conditions such as

1. \( \pi_1(X) \) is trivial
2. \( h^{0,1}(X) = 0 \)
3. \( h^{0,q}(X) = 0 \) for \( 1 \leq q \leq n - 1 \).

It is clear that 1 \( \implies \) 2 and that 3 \( \implies \) 2. For manifolds of dimension 2 (trivially) and 3, we also have that 2 \( \implies \) 3.

For the time being, let us not assume any of these conditions. In such a case, we have two families of Calabi-Yau manifolds that we will discuss. Complex tori (these are Calabi-Yau as \( ST(X) \) is a trivial vector bundle, as they are a Lie group), and K3 surfaces.

For complex tori, the Hodge diamond is given by

\[
\begin{array}{lcccc}
1 \\
2 & 2 \\
1 & 4 & 1 \\
2 & 2 \\
1 \\
\end{array}
\]

which can be computed by looking at the spaces of \((p, q)\)-forms discussed above.

For K3 surfaces, we can begin by filling in a few blanks. We know that \( h^{0,0} = h^{0,2} = h^{2,0} = h^{2,2} = 1 \), and we furthermore know (from previous lectures) that \( h^{0,1} = h^{1,0} = h^{1,2} = h^{2,1} = 0 \). Thus we begin with

\[
\begin{array}{lcccc}
1 \\
0 & 0 \\
1 & h^{1,1} & 1 \\
0 & 0 \\
1 \\
\end{array}
\]

with only one entry undetermined. However, we can compute this easily now; we know that \( b_2(X) = 22 \), and so we see immediately that the Hodge diamond for a
K3 surface is given as follows.

\[
\begin{array}{ccc}
1 \\
0 & 0 \\
1 & 20 & 1 \\
0 & 0 \\
1 
\end{array}
\]

**Calabi-Yau Threefolds** Lastly, let us briefly discuss Calabi-Yau threefolds, which are of particular interest in physics and mirror symmetry. We will in this case make the assumption that \( h^{0,1}(X) = 0 \) which implies that \( h^{0,2}(X) = 0 \). As such, the Hodge diamond—taking all symmetries into account—is given by the following.

\[
\begin{array}{ccc}
1 \\
0 & 0 \\
0 & h^{1,1} & 0 \\
1 & h^{1,2} & h^{1,2} & 1 \\
0 & h^{1,1} & 0 \\
0 & 0 \\
1
\end{array}
\]

**Remark** This is often how the Hodge diamond is presented for Calabi-Yau threefolds, and hence the confusion as to whether or not \( h^{0,0}(X) \) is at the top or the bottom. Which way is up, and which way is down?

Beyond this, we can go no further. One can examine the zoo of Calabi-Yau threefolds out there, which mathematicians and physicists have become particularly proficient at producing over the years, and there are a lot of patterns in the diagrams if you graph \( h^{1,1} - h^{1,2} \) against \( h^{1,1} + h^{1,2} \); in general, this diagram has many symmetries itself, and suggests that for any Calabi-Yau threefold \( X \) with a given \( h^{1,1} \) and \( h^{1,2} \), that there exists another Calabi-Yau threefold \( \hat{X} \) (the *mirror manifold*) with \( h^{1,1}(\hat{X}) = h^{1,2}(X) \) and \( h^{1,2}(\hat{X}) = h^{1,1}(X) \). One can interpret these equalities in terms of complex and Kähler moduli, but we will go no further into this discussion here.
3.4.3 Classifications

We will assume now that our Hodge structures satisfy $V^{p,q} = 0$ if $p, q < 0$, which naturally occurs when we are looking at the cohomology of a complex manifold.

**Hodge structures of weight 1** Recall that a Hodge structure of weight one is a decomposition

$$V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$$

with $V^{0,1} = V^{1,0}$.

**Theorem 3.19.** There is a bijective correspondence between Hodge structures of weight one and complex tori.

**Proof.** One direction of this is simple; given a complex torus, we obtain naturally a weight one Hodge structure by looking at the Hodge decomposition on $H^1(X, \mathbb{C})$.

For the other direction, consider the following maps.

$$V \rightarrow V \otimes \mathbb{C} \rightarrow V^{0,1} \oplus V^{1,0} \rightarrow V^{0,1}$$

We claim that the resulting map $V \rightarrow V^{0,1}$ embeds the former as a full-rank lattice. This follows due to the fact that $\mathbb{Z} \subset \mathbb{C}$ is invariant under complex conjugation, and so embeds diagonally in the Hodge splitting. Since the rank of $V$ is twice the complex dimension of $V^{0,1}$, the claim follows. The resulting complex torus is then $V^{0,1}/V$; it is clear that these two operations are inverse to each other. \qed

**Remark** Note that we obtained a Hodge structure of weight one in example 3.4.1. One can verify—at least in the case of a rank 2 free abelian group with almost-complex structure given by the endomorphism

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

that the resulting complex torus is $\mathbb{C}/\langle 1, i \rangle$.

**Hodge structures of weight 2** Finally, our interest is in K3 surfaces, so let us examine certain Hodge structures of weight 2.

**Definition** We say that a Hodge structure of weight 2 is of K3 type if

1. $h^{0,2} = h^{2,0} = 1$
2. $h^{p,q} = 0$ if $p < 0$ or $q < 0$. 

Note that all weight 2 Hodge structures arising from the cohomology of complex manifolds satisfy this second condition. Furthermore, note that this implies that we have
\[ V_C = V^{0,2} \oplus V^{1,1} \oplus V^{2,0} \]

**Example** Obviously, the Hodge structure on the lattice \( H^2(X, \mathbb{Z}) \) for a K3 surface \( X \) is of K3 type.

**Example** Similarly, the Hodge structure on \( H^2(A, \mathbb{Z}) \) for an abelian surface \( A \) is of K3 type.

**Example** More interestingly, we have the following. For a K3 surface \( X \), define the pairing
\[ \langle \cdot, \cdot \rangle : H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to \mathbb{Z} \]
via the formula
\[ \langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta, \]
which defines a non-degenerate inner product on this lattice.

Define now \( NS(X) = \text{im} \left( \text{Pic}(X) \to H^2(X, \mathbb{Z}) \right) \), the Néron-Severi lattice. Furthermore, define \( T = NS(X)^\perp \), the transcendental lattice. It follows then that \( T \) is of K3 type.

Note that since \( NS(X) \otimes \mathbb{C} \subset H^{1,1}(X) \) it follows that \( H^{0,2}(X) \subset T \otimes \mathbb{C} \). This is an important property for when we study the moduli of algebraic K3 surfaces.

Finally, let us discuss one last fact that will lead naturally into our discussion of the Torelli theorem. Choose any non-zero element \( \omega \in H^{0,2}(X) \). Using the pairing above, we have the following properties.

1. \( \langle \omega, \omega \rangle = 0 \)
2. \( \langle \omega, \overline{\omega} \rangle > 0 \)
3. \( \text{Span}(\omega, \overline{\omega})^\perp = H^{1,1}(X) \)

Since \( H^{0,2}(X) \) is one-dimensional, we can conclude from this that any non-zero element of \( H^{0,2} \) determines the Hodge decomposition entirely, in the following way. First, it determines \( H^{0,2}(X) \) which is one-dimensional. Second, its conjugate determines \( H^{2,0}(X) \) which is also one-dimensional. Finally, using the pairing above, we can look at the orthogonal complement, which gives us \( H^{1,1}(X) \). These properties will lead us towards the Torelli theorem, to be discussed next.
Lecture 4: Torelli theorem for complex K3 surfaces: Nadim Rustom (27/10)

In this section, we state Torelli theorem for complex K3 surfaces, and explore its applications. We mainly follow [2] and [30].

4.1 Torelli theorem: statements

Let $X$ be a complex K3 surface. Then the space $H^{1,1}(X)$ has signature $(1,19)$ so the set $\{x \in H^{1,1}(X) | (x,x) > 0\}$ has two connected components. We denote the one which contains all Kähler classes by $C_X$. This is called the positive cone. The class $d \in \text{NS}(X)$ is effective if it is represented by an effective divisor.

Suppose that we have two complex K3 surfaces $X, X'$. A Hodge isometry $\Phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ is called effective if it preserves the positive cones and induces a bijection between the respective sets of effective classes. We define the Kähler cone to be the convex subcone of the positive cone consisting of those elements which have positive inner product with any effective class in $\text{NS}(X)$.

**Exercise 4.1.** Discuss a concrete expression of the Kähler cone of a complex K3 surface in [2, VIII-Corollary 3.9]. For an algebraic K3 surface, the Kähler cone coincides with the ample cone.

**Lemma 4.2.** Let $X$ and $X'$ be two complex K3 surfaces and $\Phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ is a Hodge isometry. Then the following statements are equivalent:

- $\Phi$ is effective;
- $\Phi$ maps the Kähler cone of $X$ to the one of $X'$;
- $\Phi$ maps one element of the Kähler cone of $X$ into the Kähler cone of $X'$.

**Proof.** See [2, VIII-Proposition 3.11].

We denote the K3 lattice by $\Lambda_{K3}$. A marking on a complex K3 surface $X$ is an isometry, i.e, an isomorphism of lattices,

$$\Phi : H^2(X, \mathbb{Z}) \cong \Lambda_{K3}.$$

A marked complex K3 surface is a pair $(X, \Phi)$. The period domain of complex K3 surfaces is

$$\Omega = \{x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) | (x,x) = 0, (x, \bar{x}) > 0\}$$

which is an open subset of a 20-dimensional quadric. For any marked complex K3 surface, the period point of $(X, \Phi)$ is $\Phi_C(H^{2,0}(X)) \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$. By the Hodge-Riemann relations, the period point lies in $\Omega$. 

Theorem 4.3. (Weak Torelli theorem) Two complex K3 surfaces $X$ and $X'$ are isomorphic if and only if there are markings

$$\Phi : H^2(X, \mathbb{Z}) \cong \Lambda_{K3} \cong H^2(X', \mathbb{Z}) : \Phi',$$

whose period points in $\Omega$ coincide.

The weak Torelli theorem follows from the strong Torelli theorem.

Theorem 4.4. (Strong Torelli theorem) Let $(X, \Phi)$ and $(X', \Phi')$ be marked complex K3 surfaces whose period points on $\Omega$ coincide. Suppose that $f^* = (\Phi')^{-1} \circ \Phi$ is effective. Then there is a unique isomorphism $f : X' \to X$ inducing $f^*$.

For any $\omega \in \Omega \subset \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$, one can construct the Hodge decomposition on $\Lambda_{K3}$ whose $H^2_{\text{odd}}(X)$ is generated by $\omega$.

Theorem 4.5. (Surjectivity of the period map) Given a point $\omega \in \Omega$ inducing a decomposition on $\Lambda_{K3}$, there exists a complex K3 surface $X$ and a marking $\Phi : H^2(X, \mathbb{Z}) \to \Lambda_{K3}$ which is an isomorphism of Hodge structures.

We postpone proofs of these theorems, and we would like to discuss some applications.

### 4.2 Lattices and discriminant forms

A lattice $L$ is a free abelian group of finite rank with a symmetric non-degenerate integral bilinear form $(, ) : L \times L \to \mathbb{Z}$. We say that $L$ is even if $(x, x)$ is even for any $x \in L$. We denote the dual abelian group by $L^\vee$. The pairing gives us an injective map $L \to L^\vee, x \mapsto (y \mapsto (x, y))$. The discriminant group is $D(L) = L^\vee / L$ which is finite because the pairing is non-degenerate. For any even lattice $L$, we define the discriminant form by

$$q_L : D(L) \to \mathbb{Q}/2\mathbb{Z}, \quad x + L \mapsto (x, x) \mod 2\mathbb{Z}.$$ 

Let $l(L)$ be the minimal number of generators of $D(L)$.

**Theorem 4.6.** ([21, Corollary 1.13.3]) If a lattice $L$ is even and indefinite, and $\text{rk } L \geq l(L) + 2$, then $L$ is determined up to isometry by its rank, signature, and its discriminant form.

An embedding of lattices $L \hookrightarrow M$ is primitive if the cokernel is torsion-free.

**Theorem 4.7.** ([21, Corollary 1.12.3]) There exists a primitive embedding $L \hookrightarrow \Lambda_{K3}$ of an even lattice $L$ of rank $r$ and signature $(p, r - p)$ into the K3 lattice if $3 \leq p$, $19 \leq r - p$, and $l(L) \leq 22 - r$. 

4.3 Applications

**Exercise 4.8.** Show that there exists a polarized K3 surface of degree $2d$ of Picard rank one.

**Exercise 4.9.** Discuss the example of [30, Section 1.11].

**Exercise 4.10.** (Hassett-Tanimoto-Tschinkel-Zhang’s examples) Construct examples of two algebraic K3 surfaces $X, Y$ which are derived equivalent such that $X$ admits only finitely many automorphisms, but $Y$ admits the infinite automorphism group. In particular, one can conclude that $Y$ satisfies potential density, but we do not know whether $X$ satisfies potential density. (Hint: two K3 surfaces have isometric transcendental lattices if and only if their bounded categories of coherent sheaves are derived equivalent. Such examples are constructed in [12]. See Example 23. If you have any question about this example, just ask me (Sho). I am happy to discuss.)

4.4 Appendix by Nadim Rustom

4.4.1 Néron-Severi lattice

Identify $H^2(X, \mathbb{R})$ with the real part of $H^2(X, \mathbb{C})$. Let

$$H^{1,1}(X, \mathbb{R}) = \{ c \in H^2(X, \mathbb{R}) : (c, \omega_X) = 0 \}.$$  

The Picard group of $X$ can be identified with $H^1(X, \mathcal{O}_X^\times)$. The exponential short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{f \mapsto e^{2\pi if}} \mathcal{O}_X^\times \to 0$$

gives a long exact sequence in cohomology, which, together with the fact that $H^1(X, \mathcal{O}_X) = 0$, gives an injection

$$\delta : \text{Pic}(X) \to H^2(X, \mathbb{Z}).$$

Let $i_*H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$ be the map induced from the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{C}$.

**Theorem 4.11** (Lefschetz (1,1)-theorem). $(i_* \circ \delta)(\text{Pic}(X)) = H^{1,1}(X, \mathbb{R}) \cap \text{im} \ i_*$. 

**Proof.** This comes from the degeneration of the spectral sequence

$$H^p(X, \Omega^q_X) \Rightarrow H^{p+q}(X)$$

for $p + q = 2$. \hfill $\Box$

**Definition** The image of $\text{Pic}(X)$ by $i_* \circ \delta$ is called the Néron-Severi group of $X$ and is denoted $\text{NS}(X)$. 
Definition Let \( d \in NS(X) \). Then \( d \) is called

- effective if it comes from an effective divisor,
- irreducible if it comes from an irreducible divisor,
- nodal if it comes from a rational smooth curve.

Lemma 4.12.  
1. If \( d \in NS(X) \) and \( d \neq 0 \) and \((d,d) \geq -2\), then \( d \) is effective or \( -d \) is effective.

2. If \( d \) is irreducible then \((d,d) \geq -2\). Moreover, \( d \) is nodal if and only if \((d,d) = -2\).

Proof.  
1. The Riemann-Roch theorem for an analytic line bundle \( \mathcal{L} \) associated to a class \( d \) over a surface \( X \),

\[
h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}) + h^2(X, \mathcal{L}) = \frac{1}{12} (c_1(X)^2 + c_2(X)) + \frac{1}{2} (d, d) + (d, c_1(X)).
\]

Serre duality (and triviality of the canonical bundle) gives

\[
h^2(X, \mathcal{L}) \cong h^0(X, \mathcal{L}^{-1} \otimes K_X)
\]

and hence

\[
h^0(X, \mathcal{L}) + h^0(X, \mathcal{L}^{-1}) \geq 1 + \frac{1}{2} (d, d)^2.
\]

So if \((d,d) \geq -2\), either \( \mathcal{L} \) or \( \mathcal{L}^{-1} \) admits a global section.

2. Suppose \( d \) is irreducible. Let \( p_a \) be the arithmetic genus associated to \( d \). Since the canonical divisor is trivial, it follows by the adjunction formula that

\[
p_a = \frac{1}{2} (K_X + d, d) + 1 = 1 + \frac{1}{2} (d, d) > 0.
\]

Thus \((d,d) = -2 \Leftrightarrow p_a = 0\). \( \square \)

4.4.2 Kähler cone and Positive cone

We have seen that \( H^2(X, \mathbb{R}) \) has signature \((3, 19)\). We can write

\[
H^2(X, \mathbb{R}) = (H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})) \cap H^2(X, \mathbb{R})
\]

\[
\oplus
\]

\[
(H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R}))
\]

We have seen that \( H^{2,0}(X, \mathbb{C}) = \mathbb{C} \omega_X \) and \( H^{0,2}(X, \mathbb{C}) = \mathbb{C} \bar{\omega}_X \). Therefore

\[
(H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})) \cap H^2(X, \mathbb{R})
\]

\[\text{i.e. of genus 0.}\]
is generated by $\omega_X + \bar{\omega}_X$ and $i(\omega_X - \bar{\omega}_X)$. But by the Hodge-Riemann relations

$$(\omega_X + \bar{\omega}_X, \omega_X + \bar{\omega}_X) > 0$$

$$(\omega_X - \bar{\omega}_X, \omega_X - \bar{\omega}_X) > 0$$

so

$$(H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})) \cap H^2(X, \mathbb{R})$$

has signature (2, 0) and therefore

$$(H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R}))$$

has signature (1, 19). Thus $\{x \in H^{1,1}(X, \mathbb{R}) : (x, x) > 0\}$ has two connected components. To see this, for such an $x = (x_1, y_1, \ldots, y_{19})$, we have

$$x_1^2 - y_1^2 - \cdots - y_{19}^2 > 0$$

so the two connected components correspond to those $x$ with $x_1 > 0$ and $x_1 < 0$.

**Definition** $x \in H^{1,1}(X, \mathbb{R})$ is called a Kähler class if it comes from a closed $(1, 1)$-form associated to a hermitian metric.

The inner product of any two Kähler classes is positive. The set of Kähler classes thus lies in exactly one of the two connected components of $H^{1,1}(X, \mathbb{R})$. Moreover, if $d = [D]$ is an effective divisor and $\kappa = [\omega]$ is a Kähler class, then

$$(\kappa, d) = \int_D w > 0.$$  

**Definition** The connected component of $H^{1,1}(X, \mathbb{R})$ which contains all Kähler classes is called the positive cone and is denoted by $C_X$. The other connected component is denoted $C_X'$. The Kähler cone is the set

$$C_X^+ = \{x \in C_X : (x, d) > 0 \ \forall d \text{ effective}\}.$$  

The Kähler classes lie in the Kähler cone.

**Remark** The signature theorem with Schwarz’ inequality give us that $(x, y) \geq 0$ whenever $x, y \in \bar{C}_X$, with strict inequality whenever either $x$ or $y \in C_X$.

**Lemma 4.13.** The semigroup of effective classes on a K3 surface is generated by the nodal classes and $\bar{C}_X \cap H^2(X, \mathbb{Z})$. 

4.4 Appendix by Nadim Rustom

Proof. An irreducible class \(d \in H^2(X, \mathbb{Z})\) be associated to an effective nonzero divisor \(D\). Then either \(d\) is nodal, or \((d, d) \geq 0\). Suppose \(d\) is not nodal. Since we have \((d, \kappa) > 0\) for every Kähler class \(\kappa\), \(d\) must lie on \(\bar{C}_X\).

Conversely, if \(d\) is a nonzero integral point of \(\bar{C}_X\), then \(d\) must lie on \(\bar{C}_X\). Conversely, if \(d\) is a nonzero integral point of \(\bar{C}_X\), then \(d\) must lie on \(\bar{C}_X\).

Example [Concrete expression of the Kähler cone] Let \(\Delta = \{d \in NS(X) : (d, d) = -2\ \text{and} \ d \ \text{is effective}\}\). Then every Kähler class is contained in

\[
C_X^+ = \{y \in C_X : (y, d) > 0 \ \forall d \in \Delta\}.
\]

If \(X\) is a K3 surface, then \(C_X^+\) is the Kähler cone. To see this, recall that the Kähler cone consists of

\[
\{x \in C_X : (x, d) > 0 \ \forall \ \text{effective} \ d\}.
\]

By Lemma 4.13 it is enough to test this for nodal classes and for \(d \in \bar{C}_X^+\), but for the latter we automatically have \((x, d) > 0\) since for all \(x, y \in \bar{C}_X\), \((x, y) \geq 0\) with strict inequality if either \(x\) or \(y\) is in \(C_X\).

Any class \(\delta \in H^2(X, \mathbb{Z})\) determines an automorphic isometry \(s_\delta\) of \(H^2(X, \mathbb{Z})\), given by

\[
s_\delta(d) = d + (d, \delta)\delta.
\]

Note that \(s_\delta(\delta) = -\delta\), and \(s_\delta\) is just the reflection orthogonal to \(\delta\). We call \(s_\delta\) a Picard-Lefschetz reflection. For given \(\delta\), let \(H_\delta\) be the hyperplane of fixed points of \(s_\delta\). Then Kähler cone \(C_X^+\) is one of the connected components of \(C_X \setminus \bigcup_{\delta \in \Delta} H_\delta\). For an algebraic K3 surface, the Kähler cone coincides with the ample cone.

4.4.3 Marked K3 surfaces

Let \(X\) an \(X'\) be K3 surfaces. A Hodge isometry is an isometry

\[
\Phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})
\]

which respects the Hodge decomposition. \(\Phi\) is called effective if it is a bijection on the respective sets of effective classes.

Lemma 4.14. The following are equivalent.

1. \(\Phi\) is effective,
2. \(\Phi(C_X^+) \subset \Phi(C_{X'}^+)\),
3. \(\Phi(C_X^+) \cap C_{X'}^+ \neq \emptyset\).
4. \(\Phi\) maps the Kähler cone of \(X\) to the Kähler cone of \(X'\),
5. Φ maps at least one element of the Kähler cone of X into the Kähler cone of X'.

Proof. (i) ⇒ (ii) ⇒ (iii) is clear. We show (iii) ⇒ (i). Observe that Φ(C_X) = C_{X'}.
We only have to show that if d ∈ NS(X) is nodal, then d' = Φ(d) is effective. If
x ∈ C^+_X with Φ(x) ∈ C^+_X', then
0 < (x, d) = (Φ(x), d')
so −d' cannot be effective and hence d' is effective.

Let X be a K3 surface.

Definition A marking on X is an isometry

\[ H^2(X, \mathbb{Z}) \cong \Lambda_{K3}. \]

A K3 surface together with a marking is called a marked K3 surface \((X, \Phi)\).

Definition For \( \omega \in \Lambda_{K3} \otimes \mathbb{C} \), denote by \([\omega]\) the corresponding line and let

\[ \Omega := \{ [\omega] \in \mathbb{P}^1(\Lambda_{K3} \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \}. \]

If \((X, \Phi)\) is a marked K3 surface, then \([([\Phi \otimes \mathbb{C}](\omega_X)) \in \Omega\) by the Hodge-Riemann relations. The point \([([\Phi \otimes \mathbb{C}](\omega_X))\) is called the period point of the marked K3 surface \((X, \Phi)\).

4.4.4 Torelli theorems

We can now state the weak and strong Torelli theorems for complex K3 surfaces.

Theorem 4.15 (Weak Torelli). Let X and X' be two complex K3 surfaces. Then
X and X' are isomorphic if and only if there exist markings Φ on X and Φ' on X'
such that the period points of \((X, \Phi)\) and \((X', \Phi')\) coincide.

The weak theorem follows from the strong Torelli theorem.

Theorem 4.16 (Strong Torelli). Let \((X, \Phi)\) and \((X', \Phi')\) be two marked complex K3
surfaces whose period points coincide. Suppose further more that the Hodge isometry

\[ f^* : H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z}) \]

where \(f^* = (\Phi')^{-1} \circ \Phi\) is effective. Then there exists a unique isomorphism

\[ f : X \rightarrow X' \]

inducing \(f^*\).
4.4.5 **Surjectivity of the period map**

For any $[\omega] \in \Omega$, there exists a Hodge decomposition of $\Lambda_{K3}$ given by definition $H^{2,0}(\Lambda_{K3}) = \mathbb{C}\omega$, $H^{0,2}(\Lambda_{K3}) = \mathbb{C}\bar{\omega}$, and

$$H^{1,1}(\Lambda_{K3}) = (H^{2,0} \oplus H^{0,2})^\perp.$$ 

**Theorem 4.17** (Surjectivity of the period map). Given a point $[\omega] \in \Omega$ inducing a Hodge decomposition on $\Lambda_{K3}$, there exists a complex K3 surface $X$ and a marking $\Phi : H^2(X, \mathbb{Z}) \rightarrow \Lambda_{K3}$

which induces an isomorphism of Hodge structures.

We describe an application.

4.4.6 **Lattices and discriminant forms**

A lattice $L$ is a free abelian group of finite rank with a symmetric non-degenerate integral bilinear form $(-,-) : L \times L \rightarrow \mathbb{Z}$. We say $L$ is even if $(x,x)$ is even for all $x \in L$. Let $L^\vee = \text{Hom}(L, \mathbb{Z})$ the dual of $L$. Then the bilinear form gives an injective map

$$L \hookrightarrow L^\vee$$

$$x \mapsto (y \mapsto (x,y))$$

The discriminant group is $D(L) = L^\vee / L$. The discriminant group is finite because the pairing is non-degenerate, its order is the discriminant of $L$. We let $\ell(L)$ denote the minimum number of generators of $D(L)$. Suppose $L$ is an even lattice. We can identity $L^\vee$ with

$$\{x \in L \otimes \mathbb{Q} : (x,y) \in \mathbb{Z} \text{ for all } y \in L\}$$

and so $L^\vee$ inherits the $\mathbb{Q}$-bilinear product from $L \otimes \mathbb{Q}$. This gives the discriminant form

$$q_L : D(L) \rightarrow \mathbb{Q}/2\mathbb{Z}$$

$$x + L \mapsto (x,x).$$

**Theorem 4.18.** If a lattice $L$ is even, indefinite, and rank $L \geq \ell(L) + 2$, then $L$ is determined up to isometry by its rank, signature, and its discriminant form.

An embedding $L \hookrightarrow M$ is primitive if its cokernel is torsion-free.

**Theorem 4.19.** There exists a primitive embedding $L \hookrightarrow \Lambda_{K3}$ of an even lattice $L$ of rank $r$ and signature $(p,r-p)$ into the K3 lattice if $3 \geq p$, $19 \geq r-p$, and $\ell(L) < 22 - r$. 
4.4.7 Application

**Proposition 4.20.** There exists a polarized K3 surface of degree $2d$ of Picard rank 1.

*Proof.* Let $L = \mathbb{Z}$ be lattice of rank 1 with bilinear form determined by $(1, 1) = 2d$. Then rank $L = 1$ and $L$ has signature $(p, r - p) = (1, 0)$. We have $L^\vee = \frac{1}{2d} \mathbb{Z}$ hence $\ell(L) = 1 < 22 - r$. Therefore there exists a primitive embedding

$$L \hookrightarrow \Lambda_{K3}.$$ 

We now want to construct a Hodge structure on $\Lambda_{K3}$ such that

$$H^{1,1} \cap \Lambda_{K3} = L.$$ 

For $[\omega] \in \Omega$, let

$$\Lambda_{K3} = H^{2,0}_\omega \oplus H^{1,1}_\omega \oplus H^{0,2}_\omega$$

be the corresponding Hodge decomposition. Let $M_\omega := H^{1,1}_\omega \cap \Lambda_{K3}$. Note that if $M_\omega \cap L \neq \emptyset$ then $L \subset M_\omega$. If additionally rank $M_\omega = 1$, then $L = M_\omega$ because the embedding $L \hookrightarrow \Lambda_{K3}$ is primitive. Thus the conditions we need on $\omega$ are

1. rank $M_\omega = 1$, and
2. $L \cap M_\omega^\perp = \emptyset$.

If we get such an $\omega$, then by the surjectivity of the period map, there exists a marked complex K3 surface whose period point in $\Lambda_{K3}$ is $\omega$, and, by our construction, $NS(X) \cong L$, i.e. has Picard rank 1. It turns out that $\Phi^{-1}(x_L)$ for a generator $x_L$ of $L$ is very ample.

**Proposition 4.21.** There exists a complex K3 surface $X$ with $\text{Pic}(X)$ a rank 2 lattice with the following intersection form

$$\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$$

*Proof.* The proof follows the same method. \hfill \qed

5 Lecture 5: The weak and strong Torelli theorem: a proof

In this lecture, we study a proof of the weak and strong Torelli theorem. Our main reference is [18]. Main ingredients are

- Torelli theorem for Kummer surfaces;
• The local Torelli theorem for K3 surfaces, deformation theory;
• Projective Kummer surfaces are dense in the period domain.

This talk could be divided into two parts: Torelli theorem for Kummer surfaces and Torelli theorem for K3 surfaces.

5.1 Torelli theorem for Kummer surfaces: Dan Petersen (3/11)

The first important lemma is:

**Lemma 5.1.** [18, Corollary 3.2] Let $A$ and $A'$ be two dimensional complex tori and let $\phi^*: H^2(A',\mathbb{Z}) \rightarrow H^2(A,\mathbb{Z})$ be an Hodge isometry. Suppose that there exists an isomorphism $\psi^*: H^1(A',\mathbb{F}_2) \rightarrow H^1(A,\mathbb{F}_2)$ such that $\psi^* \wedge \psi^*$ is equals to the reduction of $\phi$ modulo two. Then $\pm \phi^*$ is induced by an isomorphism $\phi: A \rightarrow A'$.

This follows from Torelli theorem for complex tori and some elementary algebra.

Let $A$ be a complex tori and $\tilde{A}$ the blow up of $A$ along 16 two torsion points. We denote 16 exceptional divisors by $E_i$. Let $X$ be a Kummer surface associated to $A$ with the quotient map $\pi: \tilde{A} \rightarrow X$. We denote the pushforward of $E_i$ by $\bar{E}_i$. Then we have a full rank sublattice $\pi_*H^2(A,\mathbb{Z}) \oplus \bigoplus_i \mathbb{Z}\bar{E}_i \subset H^2(X,\mathbb{Z})$.

The lattice $\pi_*H^2(A,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^6$ whose intersection matrix is $2 \times$ intersection matrix of $H^2(A,\mathbb{Z})$. This lattice is primitive in $H^2(X,\mathbb{Z})$. (18 Proposition 3.5]). The pushforward $\bar{E}_i$ is a smooth nodal curve whose self intersection is $-2$. Their direct sum is not primitive in $H^2(X,\mathbb{Z})$. We denote its saturation by $K_A$.

**Exercise 5.2.** Describe this $K_A$ explicitly. See [18, Corollary 3.9].

**Proposition 5.3.** [18, Proposition 3.10] Let $A$ and $A'$ be abelian surfaces and $X$ and $X'$ corresponding Kummer surfaces. Suppose that we have a Hodge isometry $\Phi^*: H^2(X',\mathbb{Z}) \rightarrow H^2(X,\mathbb{Z})$ which maps $K_{A'}$ to $K_A$ and the positive cone to the positive cone. Then this is induced by an isomorphism $\Phi: X \rightarrow X'$.

This should follow from Lemma 5.1.

**Lemma 5.4.** [18, Lemma 4.1] Let $X$ be a K3 surface which contains 16 disjoint nodal curves $C_1, \ldots, C_{16}$ such that $\sum_i C_i$ is 2-divisible in $\text{NS}(X)$. Then $X$ is a Kummer surface associated to a complex torus $A$ such that $K_A$ is generated by $C_i$’s.

**Theorem 5.5.** (Torelli theorem for projective Kummer surfaces)[18, Theorem 4.2] Let $X$ be any K3 surface and let $X'$ be a projective Kummer surface. Suppose that we have an effective Hodge isometry $\phi^*$. Then this is induced by an isomorphism $\phi$.

**Corollary 5.6.** [18, Corollary 4.3] Let $X$ be a K3 surface and $X'$ a Kummer surface. Suppose that there is a Hodge isometry between $H^2(X,\mathbb{Z})$ and $H^2(X',\mathbb{Z})$. Then $X$ and $X'$ are isomorphic.
5.2 Deformation theory, the local Torelli theorem for K3 surfaces
: Dustin Clausen (10/11)

Here we recall some important facts from deformation theory of complex manifolds. A famous reference for this topic is [15]. A pdf file is available through our library. [31] is also useful.

Definition (A family of compact complex manifolds) Let $X$ and $B$ be complex manifolds. A family of compact complex manifolds is a proper holomorphic submersion $\phi: X \to B$. For any $t \in B$, a fiber $X_t = \phi^{-1}(t)$ is a compact complex manifold. We also call $\phi: X \to B$ as a smooth deformation of $X_t$.

We list some important facts regarding families of compact complex manifolds:

Theorem 5.7. [31, Theorem 9.3] Let $\phi: X \to B$ be a family of compact complex manifolds. Then every fiber is diffeomorphic to each other. In particular, the Betti numbers are constant.

Theorem 5.8. [31, Proposition 9.20 and Theorem 9.23] Let $\phi: X \to B$ be a family of compact complex manifolds. Suppose that for a fixed point $0 \in B$, the fiber $X_0$ is Kähler. Then there exists an open neighborhood $U$ of $0$ in $B$ such that the Hodge numbers are constant over $U$ and for any $t \in U$, $X_t$ is again Kähler.

Let $\phi: X \to B$ be a family of compact complex manifolds with $0 \in B$. Then we have the following exact sequence:

$$0 \to T_{X_0} \to T_X|_{X_0} \to T_{B,0} \otimes \mathcal{O}_{X_0} \to 0,$$

where $T_M$ is the tangent sheaf of a complex manifold $M$ and $T_{B,0}$ is the tangent space of $B$ at $0 \in B$. This gives us the coboundary map

$$\rho: T_{B,0} \to H^1(X_0, T_{X_0}).$$

This is called the Kodaira-Spencer map which can be interpreted as the classifying map of the first order deformations via $X$. (See [31] Section 9.1.2 for an explanation of this fact.)

We fix a compact complex manifold $X_0$ and consider the category of smooth deformations of $X_0$. Here a morphism between two smooth deformations $\phi: (X, X_0) \to (S, 0)$ and $\phi: (X', X_0) \to (S', 0)$ is a holomorphic map $\psi: S \to S'$ such that $\psi(0) = 0$ and $\psi^*X'$ is isomorphic to $X$ over $S$.

Let $\phi: (X, X_0) \to (S, 0)$ be a germ of smooth deformations of $X_0$. The germ $\phi$ is complete if for any other germ $\psi$ of smooth deformations of $X_0$, there exists a morphism from $\psi$ to $\phi$. When this morphism is unique, we say $\phi$ is universal.

Theorem 5.9. [15, Theorem 6.1] Let $\phi: (X, X_0) \to (S, 0)$ be a germ of smooth deformations of $X_0$. Suppose that the Kodaira-Spencer map for $\phi$ at $0 \in S$ is an isomorphism. Then $\phi$ is complete.
5.2 Deformation theory, the local Torelli theorem for K3 surfaces: Dustin Clausen (10/11)

**Theorem 5.10.** [13, Theorem 5.6] Suppose that $H^2(X_0, T_{X_0}) = 0$. Then there exists a smooth deformation $\phi : (X, X_0) \to (S, 0)$ of $X_0$ such that the Kodaira-Spencer map at $0 \in S$ is an isomorphism.

These theorems imply the existence of a complete smooth deformation under the assumption $H^2(X_0, T_{X_0}) = 0$. The existence of a complete smooth deformation for a general compact complex manifold is proved by Kuranishi. However, in general, the base space is no longer a manifold, and it is actually a complex analytic space (possibly singular). Also the Kodaira-Spencer map may not be an isomorphism. One can see $H^2(X_0, T_{X_0})$ as the obstruction space of deformations. This second cohomology vanishes for complex K3 surfaces, so a complete smooth deformation exists for complex K3 surfaces.

**Exercise 5.11.** Let $\phi : (X, X_0) \to (S, 0)$ is a smooth deformation of a complex K3 surface $X_0$. Show that there exists an open neighborhood $0 \in U \subset S$ such that for any $t \in U$, $X_t$ is a complex K3 surface.

Let $L$ be the K3 lattice. Let $\phi : \mathcal{X} \to S$ be a family of complex K3 surfaces. A marking of $\phi$ is an isomorphism of $R^2 \phi_* \mathbb{Z}$ onto the constant local system $L$ over $S$ which induces an isometry of lattices at each point in $S$. Such a marking exists when $S$ is simply connected.

**Definition (Period space)** The period space is the following smooth complex manifold:

$$\Omega = \{ [\omega] \in \mathbb{P}(L_\mathbb{C}) \mid (\omega, \omega) = 0, \ (\omega, \bar{\omega}) > 0 \}.$$  

This is 20-dimensional.

**Exercise 5.12.** Fix a point $[\omega] \in \Omega$ and consider the corresponding line $l \subset L_\mathbb{C}$. Shows that the tangent space of $\Omega$ at $[\omega]$ is naturally isomorphic to $\text{Hom}(l, l^\perp/l)$.

See [13, Chapter 6 Remark 1.1].

Suppose that we have a marked family $\phi : \mathcal{X} \to S, \alpha : R^2 \phi_* \mathbb{Z} \to L$ of complex K3 surfaces. Then we can define the period map $\tau$ mapping $t \in S$ to $\alpha(H^{2,0}(X_t)) \in \Omega$. This map is holomorphic. (See [31, Theorem 10.9].)

The differential map $d\tau(t)$ is a homomorphism from $T_{S,t}$ to $\text{Hom}(H^{2,0}(X_t), H^{1,1}(X_t))$. This map can be interpreted as follows:

**Lemma 5.13.** [13, Lemma 5.5 and Lemma 5.6] The differential map $d\tau(t)$ is the composite of the Kodaira-Spencer map and the homomorphism

$$\delta : H^1(X_t, T_{X_t}) \to \text{Hom}(H^{2,0}(X_t), H^{1,1}(X_t))$$

obtained by the cup product. Moreover $\delta$ is an isomorphism.
Corollary 5.14. [18, Corollary 5.7] Any complex K3 surface admits a smooth universal deformation. A marked deformation is universal if and only if the associated period map-germ is a local isomorphism.

5.3 A proof of Torelli theorem for complex K3 surfaces: Lars Haldvard Halle (18/11)

Let $X$ be a complex K3 surface and $\text{NS}(X) \subset H^2(X, \mathbb{Z})$ the Néron-Severi lattice $T_X$ the transcendental lattice. We call $X$ exceptional if $\text{rank} \text{NS}(X) = \dim H^{1,1}(X, \mathbb{R})$. In this situation $T_X \otimes \mathbb{C}$ is equal to $H^{2,0}(X) \oplus H^{0,2}(X)$, so $T_X \otimes \mathbb{R}$ admits a natural orientation i.e., $i\omega \wedge \bar{\omega}$ where $\omega$ is a generator for $H^{2,0}(X)$.

The first step is to show that the set of period points of projective Kummer surfaces is dense in the period domain $\Omega$.

Proposition 5.15. [18, Proposition 6.1] Let $T$ be a primitive oriented sublattice of $L$ of rank two such that $(x, x) \in 4\mathbb{Z}$ for any $x \in T$. Then there exists a marked exceptional Kummer surface $(X, \alpha : H^2(X, \mathbb{Z}) \rightarrow L)$ such that $\alpha$ maps $T_X$ isomorphically and orientation preserving onto $T$.

Proposition 5.16. [18, Proposition 6.2] The set of rationally defined 2-planes $P \subset L_{\mathbb{R}}$ satisfying $(x, x) \in 4\mathbb{Z}$ for all $x \in P \cap L$ is dense in the Grassmannian $\text{Gr}(2, L_{\mathbb{R}})$.

Corollary 5.17. [18, Corollary 6.4] The period points of marked projective Kummer surfaces lie dense in $\Omega$.

The following proposition seems to be the most technical part of the proof:

Proposition 5.18. (Burns-Rapoport lemma) [18, Proposition 7.1] Let $S$ be a complex manifold and $\phi : \mathcal{X} \rightarrow S$, $\phi' : \mathcal{X}' \rightarrow S$ two smooth families of complex K3 surfaces. Suppose that $\Phi^* : R^2\phi'_*\mathbb{Z} \rightarrow R^2\phi_*\mathbb{Z}$ is an isomorphism of local systems which induces an isometry at each point in $S$.

Suppose that there is a sequence $\{s_i\} \subset S$ converging to $s_0 \in S$ with isomorphisms $\phi_i : X_{s_i} \rightarrow X'_{s_i}$ such that $\phi_i^* = \Phi^*(s_i)$. Then $X_{s_0}$ is isomorphic to $X'_{s_0}$. Moreover if $\Phi^*(s_0)$ is an effective isometry, then there is a subsequence of $\{\phi_i\}$ which converges uniformly to an isomorphism $\phi_0 : X_{s_0} \rightarrow X'_{s_0}$ such that $\phi_0^* = \Phi^*(s_0)$.

We need one more step to conclude the proof:

Proposition 5.19. [18, Proposition 8.1] Let $S$ be a complex manifold and $\phi : \mathcal{X} \rightarrow S$, $\phi' : \mathcal{X}' \rightarrow S$ two smooth families of complex K3 surfaces. Suppose that $\phi^* : R^2\phi'_*\mathbb{Z} \rightarrow R^2\phi_*\mathbb{Z}$ is an isomorphism of local systems which induces a Hodge isometry at each point in $S$. Then the set of $s \in S$ such that $\phi^*(s)$ is effective is open.

Now the strong Torelli theorem follows. See [18, Theorem 9-1].
6 Lecture 6: Surjectivity of the period mapping: a proof: Sho Tanimoto (25/11)

In this lecture, we discuss surjectivity of the period mapping. A classical proof can be found in \[2\] VIII-14 or \[28\]. Here we follow a proof in \[13\] Chapter 7 which is very much in the spirit of the proof of Global Torelli theorem for hyperkähler varieties by Verbitsky.

6.1 Hausdorff reduction

First we recall the following result which is established in the previous section:

**Theorem 6.1.** Any two complex K3 surfaces are deformation equivalent.

**Proof.** Combine the local Torelli theorem, the density of period points of Kummer surfaces, and the fact that every Kummer surface is deformation equivalent to each other. \qed

We consider the moduli spaces of marked K3 surfaces:

\[ N = \{(X, \varphi)\}/\sim \]

where \( X \) is a complex K3 surface, \( \varphi : H^2(X, \mathbb{Z}) \to \Lambda_{K3} \) a marking, and two marked K3 surfaces \((X, \varphi)\) and \((X', \varphi')\) are equivalent if there exists an isomorphism \( g : X \to X' \) such that \( \varphi \circ g^* = \varphi' \).

The moduli space \( N \) becomes a non-Hausdorff 20-dimensional complex manifold by gluing the bases of the universal deformations \( X \to \text{Def}(X) \) for all K3 surfaces \( X \).

The local period maps glue to the global period mapping

\[ \mathcal{P} : N \to D \subset \mathbb{P}(\Lambda_{\mathbb{C}}) \]

where \( D = \{[\omega] \in \mathbb{P}(\Lambda_{\mathbb{C}}) | (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \} \). By the local Torelli theorem, the period map \( \mathcal{P} \) is a local isomorphism.

The following procedure is introduced by Verbitsky for general hyperkähler manifolds:

**Proposition 6.2** (Hausdorff reduction). There exists a complex Hausdorff manifold \( \overline{N} \) and locally isomorphic maps factorizing the period map:

\[ \mathcal{P} : N \to \overline{N} \to D, \]

such that two points on \( N \) map to the same point in \( \overline{N} \) if and only if they are inseparable points of \( N \).

A key lemma is the following:
Lemma 6.3. [13, Proposition 7.2.2] Suppose that \((X, \varphi), (X', \varphi')\) are distinct inseparable points. Then \(X \cong X'\) and \(\mathcal{P}(X, \varphi) = \mathcal{P}(X', \varphi')\) is contained in \(\alpha^\perp\) for some \(0 \neq \alpha \in \Lambda\).

For a proof of the proposition, see [14, Corollary 4.10].

6.2 Twistor lines

A subspace \(W \subset \Lambda_R\) of dimension 3 is called a positive three space if the restriction of the intersection form to \(W\) is positive definite. For any positive three space \(W\), we define its twistor line by

\[ T_W = D \cap \mathbb{P}(W_C) \]

which is a smooth conic in \(\mathbb{P}(W_C) \cong \mathbb{P}^2\). For each \([x]\) \(\in D\), one can consider the positive plane \(P(x)\) spanned by the real part and the imaginary part of \(x\). Two points \([x], [y]\) \(\in D\) are in the twistor line \(T_W\) if and only if \(P(x), P(y)\) spans \(W\).

A twistor line \(T_W\) is called generic if \(W^\perp \cap \Lambda = 0\). The twistor line \(T_W\) is generic if and only if there exists \(w \in W\) such that \(w^\perp \cap \Lambda = 0\). If \(W\) is generic, then \(x^\perp \cap \Lambda = 0\) for all \(x \in T_W\) except countably many points.

Two points \(x, y \in D\) are called equivalent if there exists a chain of generic twistor lines \(T_{W_1}, \cdots, T_{W_n}\) and points \(x = x_1, \cdots, x_{i+1} = y\) such that \(x_i, x_{i+1} \in T_{W_i}\).

Proposition 6.4. [13, Proposition 7.3.2] Any two points in \(D\) are equivalent.

We consider a ball in \(D\) and write \(B \subset \overline{B} \subset D\) which is topologically a ball in the Euclidean space.

Definition Two points \(x, y \in B \subset \overline{B} \subset D\) are called equivalent in \(B\) if there exists a chain of generic twistor lines \(T_{W_1}, \cdots, T_{W_n}\) and points \(x = x_1, \cdots, x_{n+1} = y\) such that \(x_i, x_{i+1}\) are contained in the same connected component of \(T_W \cap B\).

Proposition 6.5. Any two points \(x, y \in B \subset \overline{B} \subset D\) are equivalent in \(B\).

Proof. See [14, Proposition 3.10].

6.3 Hyperkähler structures on K3 surfaces

Here we recall a definition of hyperkähler manifolds:

Definition A compact Kähler manifold is hyperkähler if it is simply connected and there is a nondegenerate holomorphic symplectic two form.

Example Complex K3 surfaces are examples of hyperkähler manifolds.

Example Let \(S\) be a complex K3 surface. Then the Hilbert scheme of points on \(S\) \(\text{Hilb}^{[r]}(S)\) is a higher dimensional example of hyperkähler manifolds.
Theorem 6.6. Let $X$ be a hyperkähler manifold with a Kähler class $\alpha \in H^2(X, \mathbb{Z})$. Let $M$ be the underlying differentiable manifold of $X$ with the complex structure $I \in \text{End}(TM)$. Then there exists a Kähler metric $g$ and complex structures $J, K$ such that

1. The metric $g$ is Kähler with respect to complex structures $I, J, K$,
2. The Kähler form $\omega_I = g(I, \cdot)$ satisfies $\alpha = [\omega_I]$,
3. The complex structures $I, J, K$ satisfy the standard Hamiltonian relations:
   $$I^2 = J^2 = K^2 = IKJ = -1.$$

Proof. Essential ingredients are Yau’s solutions of Calabi-Yau conjecture, Bochner’s theorem, Newlander-Nirenberg theorem, the computation of the holonomy group. See [22, Section 3.2].

Remark Conversely if a simply connected compact manifold $M$ has such a hamiltonian structure which is compatible with a Kähler metric, then $M$ admits a hyperkähler structure.

For each $(a, b, c) \in S^2$, $\lambda = aI + bJ + cK$ is a complex structure on $M$ with respect to $g$ which is Kähler. Thus we obtain a differentiable family of compact hyperkähler manifolds $(M, \lambda)$ over $S^2 \cong \mathbb{P}^1$. One can make this as a holomorphic family. (See [13, Section 7.3.2].) We call it as the twistor family $X(\alpha) \rightarrow T(\alpha) \cong \mathbb{P}^1$ associated to $\alpha$.

Suppose that $(X, \varphi)$ is a marked K3 surface with a Kähler class $\alpha$. Then we have the period mapping

$$\mathcal{P} : T(\alpha) \rightarrow D \subset \mathbb{P}(\Lambda_C).$$

In fact, the period map identified $T(\alpha)$ with the twistor line $T_{W_\alpha}$ associated to the positive tree space $W_\alpha = \varphi(\mathbb{R} \alpha \oplus H^{1,1}(X, \mathbb{R})\perp)$.

6.4 A proof

We established the following result in the previous section:

Lemma 6.7. Let $X$ be a complex K3 surface with $\text{Pic}(X) = 0$. Then any nonzero class in the positive cone $C_X^+$ is Kähler class.

Proposition 6.8. [13, Proposition 7.3.9] Consider a marked K3 surface $(X, \varphi) \in N$ and assume that its period $\mathcal{P}(X, \varphi)$ is contained in a generic twistor line $T_W \subset D$. Then there exists a unique lift of $T_W$ to a curve in $\overline{N}$ through $(X, \varphi)$. 

Theorem 6.9. [13, Theorem 7.4.1] Let $N^\circ$ be a connected component of $N$. Then the period mapping

$$\mathcal{P} : N^\circ \to D$$

is surjective.

Working with local properties more, one can prove the following theorem:

**Theorem 6.10.** [13, Proposition 7.4.3] The period map induces a covering space $\mathcal{P} : \overline{N} \to D$.

Since $D$ is simply connected, each connected component of $\overline{N}$ is isomorphic to $D$. Moreover one can show that there are at most two connected components which are interchanged by $-1$.

7 Lecture 7: Moduli spaces of polarized K3 surfaces: Dan Petersen (2/12)

The goal of this section is to construct the coarse moduli space of polarized K3 surfaces of degree 2$d$. We follow the treatment in [13] using Torelli theorem for K3 surfaces.

### 7.1 Moduli functor

Let $S$ be a noetherian scheme (e.g., $\text{Spec}(\mathbb{C})$, $\text{Spec}(\mathbb{Q})$, or $\text{Spec}(\mathbb{Z})$.) We fix a positive integer $d$. We consider the moduli functor:

$$K_d : (\text{Sch}/S) \to (\text{Sets}),$$

such that for each scheme $T$ of finite type over $S$, we have

$$K_d(T) = \{(f : \mathcal{X} \to T, \mathcal{L})\}/\sim$$

where $f : \mathcal{X} \to T$ is a smooth proper morphism and $\mathcal{L}$ a line bundle on $\mathcal{X}$ such that for any geometric point $\text{Spec}(k) \to T$, the base change yields a K3 surface $X_k$ with a primitive ample line bundle $L_k$ with $(L_k, L_k) = 2d$. Two such pairs $(f, \mathcal{L}, (f', \mathcal{L}')$ are equivalent if there exist a $T$-isomorphism $\psi : \mathcal{X} \cong \mathcal{X}'$ and a line bundle $L_0$ on $T$ such that $\psi^* \mathcal{L}' \cong \mathcal{L} \otimes f^* L_0$.

Unfortunately this functor is not represented by any $S$-scheme (but by a DM-stack.) Hence we consider the notion of the coarse moduli space:

The coarse moduli space $K_d$ is a $S$-scheme with a functor transformation:

$$\Phi : K_d \to \overline{K}_d =: h_{K_d}$$

such that
• For any algebraically closed field $k$, the induced map $K_d(k) \to K_d(k)$ is bijective;

• For any $S$-scheme $N$ with a natural transformation $\Psi : K_d \to N$, we have a unique $S$-morphism $\pi : K_d \to N$ such that $\Psi = \pi \circ \Phi$.

Our goal of this section is to prove the following theorem:

**Theorem 7.1.** For $S = \text{Spec}(\mathbb{C})$, the moduli functor $K_d$ is coarsely represented by a quasi projective variety $K_d$.

### 7.2 Hilbert schemes

Let $(X, L)$ be a polarized K3 surface of degree $2d$. Its Hilbert polynomial is $P(t) = dt^2 + 2$ by Riemann-Roch.

**Theorem 7.2.** [13, Theorem 2.2.7] Let $X$ be a complex K3 surface and $L$ an ample line bundle on $X$. Then $L^k$ is very ample for $k \geq 3$.

Thus if we have a polarized K3 surface $(X, L)$ of degree $2d$, then we can embed $X$ into the fixed projective space $\mathbb{P}^N$ whose the Hilbert polynomial is $P(3t)$.

Consider the Hilbert scheme:

$$\text{Hilb} := \text{Hilb}_{\mathbb{P}^N}^{P(3t)}.$$  

This comes with the universal family $Z \to \text{Hilb}$ which is flat.

**Proposition 7.3.** [13, Proposition 5.2.1] There exists a subscheme $H \subset \text{Hilb}$ with the following universal property: A morphism $T \to \text{Hilb}$ factors through $H$ if and only if the pullback $Z_T \to T$ satisfies the following property:

1. The morphism $f : Z_T \to T$ is smooth such that all fibers are K3 surfaces;

2. Let $p : Z_T \to \mathbb{P}^N$ be the natural projection, then

$$p^*\mathcal{O}(1) \cong L^3 \otimes f^*L_0,$$

for some $L \in \text{Pic}(Z_T)$ and $L_0 \in \text{Pic}(T)$;

3. The above line bundle $L$ is primitive on each geometric fiber;

4. Restriction yields isomorphisms

$$H^0(\mathbb{P}^N_{k(s)}, \mathcal{O}(1)) \cong H^0(Z_s, L^3_s),$$

for all fibers $Z_s$ of $f$.

Note that the linear algebraic group $\text{PGL}_{N+1}$ acts on $\mathbb{P}^N$, $\text{Hilb}$, and $H$. The set of $\text{PGL}$-orbits on $H(\mathbb{C})$ is the set of isomorphism classes of polarized K3 surfaces of degree $2d$. 
7.3 Moduli spaces via the global period domain

We recall that the K3 lattice is

$$\Lambda_{K3} = U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

Let $l = e + df \in U$ where $e, f$ is a standard basis for $U$ such that $e^2 = f^2 = 0, e \cdot f = 1$. Then $l^2 = 2d$ and this $l$ is unique up to automorphisms of $\Lambda$ by Nikulin. We define

$$\Lambda_d = \langle l \rangle \cong \langle -2d \rangle \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

Also we define the period domain:

$$D_d = \{ [\omega] \in \mathbb{P}(\Lambda_d \otimes \mathbb{C}) \mid (\omega, \omega) = 0, \ (\omega, \bar{\omega}) > 0 \}^+$$

where $+$ denotes one of connected components.

We consider the orthogonal group $O(\Lambda_d)$. We denote the index two subgroup of $O(\Lambda_d)$ preserving the connected component $D_d$ by $O^+(\Lambda_d)$. We have a natural homomorphism:

$$O^+(\Lambda_d) \to O(D(\Lambda_d)),$$

where $D(\Lambda_d)$ is the discriminant form of $\Lambda_d$. We denote the kernel of this homomorphism by $\tilde{O}^+(\Lambda_d)$, and we call it the stable orthogonal group of $\Lambda_d$. By Nikulin, any element of $\tilde{O}^+(\Lambda_d)$ can be lifted to an orthogonal element of $O^+(\Lambda_{K3})$ which preserves $l$.

We define the global period domain:

$$\mathcal{F}_d = \tilde{O}^+(\Lambda_d) \backslash D_d.$$

An important tool to study this type of orthogonal modular spaces is the Baily-Borel compactification:

**Theorem 7.4.** [T] The space $\mathcal{F}_d$ is a quasi-projective variety.

Let $H$ be the subscheme in Hilb which we constructed in the previous section. It comes with the universal family $\mathcal{Z}_H \to H$, thus we have the period map which is a holomorphic map:

$$H(\mathbb{C}) \to \mathcal{F}_d = \tilde{O}^+(\Lambda_d) \backslash D_d.$$

By the local Torelli theorem, the image is an open set in analytic topology, and set-theoretically, it is equal to $\text{PGL}(H(\mathbb{C}))$ because of Torelli theorem.

**Theorem 7.5.** [T] A holomorphic map $H(\mathbb{C}) \to \mathcal{F}_d$ is algebraic.

Thus we realized $\text{PGL}(H(\mathbb{C}))$ as a Zariski open subset of a quasi-projective variety $\mathcal{F}_d$, and we denote this variety by $K_d$.

**Corollary 7.6.** [13, Corollary 6.4.3] The variety $K_d$ is a coarse moduli space of polarized K3 surfaces of degree $2d$ which is an irreducible variety of dimension 19.

If one considers the moduli functor of polarized K3 surfaces with some level structures, then this functor can be represented by a smooth variety and it comes with the universal family. See the discussion of [13, Section 6.4.2].
8 Lecture 8: Kodaira dimension of moduli spaces: Sho Tanimoto (9/12)

In this section, our ground field is \(\mathbb{C}\). We denote the coarse moduli space of polarized K3 surfaces of degree \(2d\) over \(\mathbb{C}\) by \(\mathcal{K}_d\). In this section, we would like to discuss the birational type of the moduli space \(\mathcal{K}_d\).

8.1 Unirationality

The moduli space \(\mathcal{K}_d\) tends to be unirational when \(d\) is relatively small.

**Theorem 8.1** (Mukai). *The moduli space \(\mathcal{K}_d\) is unirational if \(1 \leq d \leq 10\) or \(d = 12, 17, 19\).*

*Proof.* When \(d = 1, 2, 3, 4\), this is classical. Let us prove our assertion for \(d = 2\), i.e., the moduli space of K3 surfaces of degree 4. Let \(\mathbb{P}^N\) be the space of quartic surfaces in \(\mathbb{P}^3\) and let \(U \subset \mathbb{P}^N\) be the Zariski open subset parametrizing smooth quartic surfaces. Then we have the natural map \(U \to \mathcal{K}_2\). We claim that this map is dominant. Indeed, for a very general K3 surface \((X, L)\) of degree 4, we have \(\text{Pic}(X) = \mathbb{Z}L\). This implies that \(L\) is very ample. (See [13, Example 2.3.9].) Thus the complete linear system of \(L\) defines an embedding into \(\mathbb{P}^3\), so \((X, L)\) is a quartic surface.

For \(d = 1, 3, 4\), we have an explicit way to describe a very general K3 surface of degree \(2d\). (They are double covers of the plane, or the complete intersections in the projective space.) These description can be used to prove that \(\mathcal{K}_d\) is unirational.

For other \(d\) listed above, Mukai showed how to obtain a very general K3 surface of degree \(2d\), e.g., the intersection of a linear space with a homogeneous space such as a Grassmanian.

Note that any unirational variety has negative Kodaira dimension.

8.2 Kodaira dimension

The moduli space \(\mathcal{M}_g\) of curves of genus \(g\) and the moduli space \(\mathcal{A}_g\) of principally polarized abelian varieties of dimension \(g\) also tend to be unirational when \(d\) is small. However, it is shown by Mumford that these moduli spaces are actually of general type when \(g\) is sufficiently large. It is proved by Grissenko, Hulek, and Sankaran that the same thing applies to \(\mathcal{K}_d\):

**Theorem 8.2.** [9, Theorem 1] *The moduli space \(\mathcal{K}_d\) is of general type if \(d > 61\) or \(d = 46, 50, 54, 57, 58, 60\). The Kodaira dimension of \(\mathcal{K}_d\) is non-negative if \(d \geq 40\) and \(d \neq 41, 44, 45, 47\).*
The rest of this section is devoted to some aspects of the proof of this theorem. The strategy of the proof of this theorem was applied to other moduli spaces which are orthogonal modular varieties, e.g., the moduli spaces of polarized hyperkähler manifolds by Grissenko, Hulek, and Sankaran as well as the moduli spaces of special cubic fourfolds by Tanimoto and Várilly-Alvarado \cite{25}.

8.3 Low weight cusp form trick

Let $L$ be a lattice of signature $(2, n)$ where $n > 1$. We consider the period domain $D_L$ which is one of connected components of

\[ \{ [x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0, \ (x, \bar{x}) > 0 \}. \]

Let $O^+(L)$ be the subgroup of the orthogonal group of $L$ preserving $D_L$. Suppose that $\Gamma \subset O^+(L)$ be a finite index subgroup. We define the orthogonal modular variety by

\[ F_L(\Gamma) := \Gamma \backslash D_L, \]

which is a quasi-projective variety by \cite{1}.

A modular form of weight $k$ with a character $\chi : \Gamma \to \mathbb{C}^\times$ is a holomorphic function $F : D_L^* \to \mathbb{C}$ where $D_L^*$ is the affine cone over $D_L$ such that

1. $F(tZ) = t^{-k}F(Z)$ for any $t \in \mathbb{C}^\times$;

2. $F(gZ) = \chi(g)F(Z)$ for any $g \in \Gamma$.

A modular form $F$ is a cusp form if it vanishes at every cusp. We denote the space of modular forms and cusp forms by $M_k(\Gamma, \chi), S_k(\Gamma, \chi)$ respectively.

**Theorem 8.3.** \cite{9} **Theorem 1.1** Suppose that $L$ is a lattice of signature $(2, n)$ where $n \geq 9$ and $\Gamma$ is a finite index subgroup of $O^+(L)$. Assume that there is a non-zero cusp form $F_a$ of weight $a < n$ with a character $\chi \in \{1, \det\}$ such that $F_a$ vanishes along the ramification divisors of the projection

\[ \pi : D_L \to F_L(\Gamma). \]

Then $F_L(\Gamma)$ is of general type.

If $S_n(\Gamma, \det) \neq 0$, then the Kodaira dimension of $F_L(\Gamma)$ is non-negative.

**Sketch of the proof.** It takes the following steps:

- First they construct a projective toroidal compactification $\overline{F}_L(\Gamma)$ of $F_L(\Gamma)$ with only canonical singularities and no ramification divisors at the infinity. (See \cite{9} Theorem 2.1.)
Suppose that \( F_{nk} \in M_{nk}(\Gamma, \chi) \). Let \( dZ \) be a holomorphic volume element on \( D_L \). Then the differential form \( \Omega(F_{nk}) := F_{nk}(dZ)^k \) is \( \Gamma \)-invariant and determines a section of the pluri canonical bundle \( kK_{\mathcal{F}_L(\Gamma)} \) away from the branch locus of \( \pi \) and cusps;

To extend the differential form \( \Omega(F_{nk}) \) to \( \mathcal{F}_L(\Gamma) \), one needs to discuss three types of obstructions; elliptic obstructions, reflective obstructions, and cusp obstructions;

Suppose that we have a cusp form \( F_a \) of small weight \( a < n \). Let \( k \) be a sufficiently divisible positive integer. We consider \( F_{0nk}^a = F_k^a F_{n-a}^k \) where \( F_{(n-a)k} \in M_{(n-a)k}(\Gamma) \). Then \( \Omega(F_{0nk}^a) \) extends to \( \mathcal{F}_L(\Gamma) \).

Now \( F_k^a M_{(n-a)k}(\Gamma) \) is a subspace of \( H^0(\mathcal{F}_L(\Gamma), kK_{\mathcal{F}_L(\Gamma)}) \), and \( M_{(n-a)k}(\Gamma) \) grows like \( k^n \).

\[ \square \]

### 8.4 Borcherds form

Our goal in this section is to prove that the global period domain \( \mathcal{F}_d \) is of general type if \( d \) is sufficiently large. To do this, we use low weight cusp form trick so that we only need to construct a single cusp form of low weight with certain properties. We use the Borcherds form.

Let \( L_{2,26} = 2U \oplus 3E_8(-1) \) be an even unimodular lattice of signature \((2,26)\). Then \( M_{12}(O^+(L_{2,26}), \det) \) is one dimensional spanned by \( \Phi_{12} \) which is called the Borcherds form. (\[5\]) The zeros of \( \Phi_{12} \) lie on rational quadratic divisors of \((-2)\)-vectors, i.e., \( \Phi_{12}(Z) = 0 \) if and only if there exists \( r \in L_{2,26} \) such that \( r^2 = -2 \) and \( (r, Z) = 0 \). Moreover, the multiplicity of these rational quadratic divisors is one.

Choose \( l \in E_8(-1) \) such that \( l^2 = -2d \). We embed \( \Lambda_d \) into \( L_{2,26} \) by sending \( \langle -2d \rangle \) to \( \langle l \rangle \) and \( 2U \oplus 2E_8(-1) \) to \( 2U \oplus 2E_8(-1) \) in the obvious way. Let

\[ R_l = \{ r \in E_8(-1) \mid r^2 = -2, \quad (r, l) = 0 \} \]

and \( N_l = \#R_l \). The function

\[ F_l = \frac{\Phi_{12}(Z)}{\prod_{r \in R_l/\{\pm 1\}}(Z, r)} \bigg|_{D_d} \]

is called the quasi pullback of the Borcherds form, and this is a modular form of weight \( 12 + N_l/2 \) for \( \tilde{O}^+(\Lambda_d) \) and \( \det \). Moreover if \( N_l > 0 \), then \( F_l \) is a cusp form and vanishes along the ramification divisors of the projection \( \pi : D_d \to \mathcal{F}_d \). (See \[9\], Theorem 6.2.)

Hence to prove that \( \mathcal{F}_d \) is of general type for sufficiently large \( d \), we only need to find \( l \in E_8 \) such that \( l^2 = 2d \) and \( 0 < N_l < 7 \).
Theorem 8.4. [9, Theorem 7.1] Such a vector \( l \in E_8 \) exists if
\[
4N_{E_7}(2d) > 28N_{E_6}(2d) + 63N_{D_6}(2d),
\]
holds where \( N_L(2d) \) is the number of representations of \( 2d \) by the lattice \( L \).

The left hand side has the asymptotic of order \( d^{5/2} \) and the right hand side has the asymptotic of order \( d^2 \), so this inequality holds for sufficiently large \( d \). Thus our assertion follows.

9 Lecture 9: K3 surfaces of geometric Picard rank 1 defined over number fields: Dino Destefano (11/2)

So far we have discussed the geometry of K3 surfaces, i.e., their topology, Hodge theory, Torelli theorems, and moduli spaces. From now on we focus more on the arithmetic of K3 surfaces defined over non-closed fields.

Our first topic in the arithmetic is the Picard group of K3 surfaces, e.g., how to compute its geometric Picard rank and its lattice structures. It turns out that this is quite nontrivial task even for some explicit K3 surfaces such as smooth quartic surfaces.

9.1 Results by Ellenberg

In this lecture, we discuss results of [8]. Let \( d \) be a positive integer and \( K_d \) the moduli stack of polarized K3 surfaces of degree \( 2d \) over \( \bar{\mathbb{Q}} \). We denote its the coarse moduli space by \( K_d \). Then it follows from Hodge theory that for a very general point in \( K_d(\mathbb{C}) \), i.e., any point outside a countable union of subvarieties, the corresponding K3 surface has Picard rank one. However, it is unclear if there is a polarized K3 surface of degree \( 2d \) defined over \( \bar{\mathbb{Q}} \) whose Picard rank is one. Jordan Ellenberg corrected this embarrassing situation in [8].

Theorem 9.1. [8, Theorem 1] For each \( d > 0 \), there exists a polarized K3 surface of degree \( 2d \) defined over \( \bar{\mathbb{Q}} \) whose geometric Picard rank is one.

Proof. Key ingredients are \( \ell \)-adic Galois representations, the Hilbert irreducibility theorem, and fine moduli spaces of polarized K3 surfaces with level structures. See [8].

10 Lecture 10: Examples by van Luijk : Fabien Pazuki (16/2)

Ellenberg showed that there exists a K3 surface defined over some number field \( K \) with the geometric Picard rank one. A natural question is how about over \( \mathbb{Q} \)? Can
we find an explicit equation over $\mathbb{Q}$ defining a K3 surface with the geometric Picard rank one?

This question had been open for a while, and then was corrected by van Luijk in [29]. In this lecture we discuss his results. We follow his paper [29] as well as the exposition in [30, Section 2].

Let us state van Luijk's result here: Define the following homogeneous equations:

\[
\begin{align*}
  f_1 &= x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw + y^3 \\
    &\quad + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \\
  f_2 &= xy^2 + xyz - xz^2 - yz^2 + z^3, \\
  g_1 &= z^2 + xy + yz, \\
  g_2 &= z^2 + xy.
\end{align*}
\]

Let $h$ be a homogeneous polynomial of degree 4 with $\mathbb{Z}$-coefficients. We define the quartic surface $X_h$ defined over $\mathbb{Q}$ in $\mathbb{P}^3$ by

\[wf_1 + 2zf_2 = 3g_1g_2 + 6h.\] (1)

Then van Luijk showed:

**Theorem 10.1.** [29, Theorem 3.1] For any $h \in \mathbb{Z}[x,y,z,w]_4$, the surface $X_h$ is smooth over $\mathbb{Q}$ and has the geometric Picard rank one.

### 10.1 Specializations, reduction modulo $p$

If we have an integral model $X$ of a variety $X$ defined over a number field $K$, then one can take the reduction modulo $p$. This defines a natural homomorphism:

\[\text{NS}(X) \leftarrow \text{NS}(X) \rightarrow \text{NS}(X_p)\]

One can use this to define the specialization map:

**Theorem 10.2.** [19, Proposition 3.3] Let $R$ be a discrete valuation ring with the fraction field $K$ and the residue field $k$. Fix an algebraic closure $\bar{K}$ of $K$. Choose a non-zero prime ideal $p$ of the integral closure $\bar{R}$ of $R$ in $\bar{K}$, so $\bar{k} = \bar{R}/\bar{p}$ is an algebraic closure of $k$. Let $X$ be a smooth proper $R$-scheme. Then we have a natural homomorphism:

\[s_p : \text{NS}(X_{\bar{K}}) \rightarrow \text{NS}(X_{\bar{k}})\]

**Theorem 10.3.** [19, Proposition 3.3] We use notations in Theorem 10.2. Let $p = \text{char}(k)$.

1. If $p = 0$, then $s_p$ is injective and cokernel is torsion-free;
2. If $p > 0$, then (1) holds after tensoring by $\mathbb{Z}[1/p]$;
3. In any case, we have $\text{rank } \text{NS}(X_{\bar{K}}) \leq \text{rank } \text{NS}(X_{\bar{k}})$.
10.2 The cycle class map

Let $X$ be a smooth projective variety defined over a finite field $\mathbb{F}_q$ where $q = p^r$. We denote its base change $X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ by $\bar{X}$. We consider its étale site $\bar{X}_{\text{ét}}$ and $\ell$ be a prime different from $p$. Then we have the exact sequence of group schemes:

$$0 \to \mu_\ell^\ast \to \mathbb{G}_m \to \mathbb{G}_m \to 0.$$  

The long exact sequence in étale cohomology gives us a boundary mapping

$$\delta : \text{Pic}(\bar{X}) = H^1_{\text{ét}}(\bar{X}, \mathbb{G}_m) \to H^2_{\text{ét}}(\bar{X}, \mu_\ell^\ast).$$

Taking the inverse limit, we obtain

$$\delta : \text{Pic}(\bar{X}) \to H^2_{\text{ét}}(\bar{X}, \mathbb{Z}_\ell(1)).$$

This gives rise an injective homomorphism

$$c : \text{NS}(\bar{X}) \otimes \mathbb{Q}_\ell \hookrightarrow H^2_{\text{ét}}(\bar{X}, \mathbb{Z}_\ell(1)),$$

which is compatible with the Galois action of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

**Proposition 10.4.** [22, Corollary 2.3] Write $\sigma \in \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ the Frobenius element $x \mapsto x^q$. Let $\sigma^\ast(0)$ be the induced automorphism on $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$. Then the geometric Picard rank $\rho(\bar{X})$ is bounded by the number of eigenvalues of $\sigma^\ast(0)$, counted with multiplicity, of the form $\zeta/q$ where $\zeta$ is a roof of unity.

**Remark** The Tate conjecture predicts that for any finite extension $\mathbb{F} \subset \bar{\mathbb{F}}_q$ over $\mathbb{F}_q$, we have

$$c(\text{NS}(X_\mathbb{F}) \otimes \mathbb{Q}_\ell) = H^2_{\text{ét}}(\bar{X}, \mathbb{Z}_\ell(1))^{\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F})}.$$ 

In particular, the bound in Proposition 10.4 is equal to the geometric Picard rank of $X$. Tate conjecture is known for K3 surfaces when $q$ is odd. We will come back to this point in the future.

**Remark** According to Tate conjecture, the Picard rank of $X$ must be even when $X$ is a K3 surface.

10.3 A proof of the main theorem

The proof of the main theorem takes the following steps:

Let $X_h$ be the integral model defined by the equation [1]

1. Note that $X_h \otimes \mathbb{F}_2$ and $X_h \otimes \mathbb{F}_3$ are smooth:
2. For \( p = 2, 3 \), show that the bound in Proposition 10.4 is 2. This can be done by using point counting and Lefschetz fixed points formula. See [29, Section 3] or [30, Section 2.4]. In particular, this shows that \( \rho(X_h) \) is at most 2.

3. Find generators for \( \text{NS}(X_{h,2}) \) and \( \text{NS}(X_{h,3}) \), and show that square classes of \( \det(\text{NS}(X_{h,2})) \) and \( \det(\text{NS}(X_{h,3})) \) are different.

4. We conclude that \( \rho(X_h) = 1 \).

For more details, see [29, Section 3].

11 Lecture 11: Algorithms to compute the geometric Picard rank: Oscar Marmon (1/3)

In this lecture, we discuss an algorithm to compute the geometric Picard rank of algebraic K3 surfaces defined over number fields. There are three proposed algorithms. One in [23] can be applied to any smooth projective varieties. Another is provided in [11], and this only applies to degree 2 K3 surfaces. Here we discuss an algorithm in [6] relied on the Hodge conjecture for the products of K3 surfaces. Let us state his results:

**Theorem 11.1.** [6, Theorem 5] There exists an algorithm which, given a K3 surface \( X \) over a number field, either it returns its geometric Picard rank or does not terminate. Moreover, if \( X \times X \) satisfies the Hodge conjecture for codimension 2 cycles, then the algorithm applied to \( X \) terminates.

A key ingredient is [6, Theorem 1]. Let \( X \) be a K3 surface defined over a number field \( k \). Let \( T_X \) be the orthogonal complement of \( \text{NS}(X_\mathbb{C}) \) in \( H^2(X(\mathbb{C}), \mathbb{Q}) \). This is a sub rational Hodge structure of \( H^2(X(\mathbb{C}), \mathbb{Q}) \). Let \( E \) be the endomorphism algebra of \( T_X \) as a rational Hodge structure. It is shown in [32] that \( E \) is either a totally real field or a CM field.

**Theorem 11.2.** [6, Theorem 1]

1. If \( E \) is a CM field or the dimension of \( T_X \) as an \( E \)-vector space is even, then there exists infinitely many finite places \( p \) of good reduction that \( \rho(X_p) = \rho(X) \).

2. Assume that \( E \) is a totally real field and the dimension of \( T_X \) as an \( E \)-vector space is odd. Let \( p \) be a finite place of \( k \) of good reduction. Assume that the characteristic of \( p \) is at least 5. Then

\[
\rho(X_p) \geq \rho(X) + [E : \mathbb{Q}].
\]

Moreover there are infinitely many \( p \) of good reduction such that the equality holds.
11.1 Mumford-Tate groups

Let $S$ be the Deligne torus, i.e., the algebraic group over $\mathbb{R}$ defined as

$$S = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m).$$

Let $H$ be a finite dimensional vector space over $\mathbb{Q}$. Giving a Hodge structure to $H$ is equivalent to giving an action of $S$ on $H \otimes \mathbb{R}$.

**Definition** Let $H$ be a rational Hodge structure. The Mumford-Tate group of $H$ is the smallest algebraic subgroup $\text{MT}(H)$ of $\text{GL}(H)$ such that $\text{MT}(H)_{\mathbb{R}}$ contains the image of $S$ in $\text{GL}(H_{\mathbb{R}})$.

Next we consider the $\ell$-adic theory. Let $k$ be a number field. Let $X$ be a smooth projective geometrically integral variety over $k$. Then we can consider the $\ell$-adic Galois representation:

$$\rho_{\ell} : G_k := \text{Gal}(\bar{k}/k) \to \text{GL}(H^i_{\text{ét}}(X, \mathbb{Q}_\ell)).$$

**Definition** Let $G_\ell$ be the Zariski closure of the image of $G_k$ in the algebraic group $\text{GL}(H^i_{\text{ét}}(X, \mathbb{Q}_\ell))$. This is called the algebraic monodromy group associated to $\rho_{\ell}$.

The relation between the Mumford-Tate group and the algebraic monodromy group is given by the following conjecture:

**Conjecture 11.3** (Mumford-Tate conjecture). With notations above, let $G_\ell^0$ be the identity component of $G_\ell$. Then we have

$$G_\ell^0 \cong \text{MT}(H^i(X_\mathbb{C}, \mathbb{Q}_\ell)).$$

For K3 surfaces, this conjecture is known by Tankeev [26] and [27].

Let $X$ be an algebraic K3 surface defined over $\mathbb{C}$, and consider the rational Hodge structure $H = H^2(X(\mathbb{C}), \mathbb{Q})$. Then we have the following orthogonal decomposition:

$$H = \text{NS}(X)_{\mathbb{Q}} \oplus T_X.$$ 

Then $T_X$ is the smallest rational Hodge structure containing $H^2(X, \mathcal{O}_X)$, and it comes with the restriction of the cup product which gives a polarization $\psi$ on $T_X$. Since $\text{NS}(X)$ is spanned by Hodge classes, we can identify the Mumford-Tate groups of $H$ and $T_X$. Since $\psi$ is a polarization, $\text{MT}(T_X)$ is a subgroup of the orthogonal similitudes $\text{GO}(T_X, \psi)$.

Let $E$ be the endmorphisms algebra of $T_X$. Zarhin showed the following theorem in [32]:

**Theorem 11.4.** [32, Theorem 2.2.1] The Mumford-Tate group of $T_X$ is the centralizer of $E$ in $\text{GO}(T_X, \psi)$. 
The following Corollary follows from the above theorem as well as the Mumford-Tate conjecture:

**Corollary 11.5.** Let $X$ be a K3 surface defined over a number field $k$. Let $G_\ell$ be the algebraic monodromy group associated to $T_{X,\ell}$. Then the identity component $G_\circ_\ell$ is the centralizer of $E \otimes \mathbb{Q}_\ell$ in the group of orthogonal similitudes $GO(T_{X,\ell}, \psi_\ell)$.

The proof of Theorem 11.2 follows from this Corollary. See Section 3 of [6].

### 11.2 Applications: infinitely many rational curves on K3 surfaces

Theorem 11.2 has some interesting applications in the geometry of K3 surfaces. Let $X$ be a K3 surface over an algebraically closed field $K$. Then it is shown by Mori-Mukai that $X$ contains a rational curve. Moreover they showed that for a very general K3 surface $X$, $X$ has infinitely many rational curves. To prove this, first we specialize $X$ to a K3 surface $X_0$ of higher Picard rank. Then the polarization can be expressed as a sum of linearly independent rational curves. This union of rational curves deforms to an irreducible rational curve.

However, Mori-Mukai’s arguments require $X$ to be very general. Bogomolov-Hassett-Tschinkel and Li-Liedtke came up with a mixed-characteristic version of Mori-Mukai’s arguments, and they showed the following striking theorem:

**Theorem 11.6.** [4] and [17] Let $X$ be a K3 surface whose geometric Picard rank is odd. Then $X$ contains infinitely many rational curves.

It is shown by Bogomolov and Tschinkel that K3 surfaces with elliptic fibrations or the infinite automorphism group has infinitely many rational curves. In particular, these properties hold for K3 surfaces of Picard rank $\geq 5$. So now only remaining cases are K3 surfaces of Picard rank 2 and 4.

A key point of [4] and [17] is to produce many rational curves at specializations. To do this, they need infinitely many primes $p$ such that

$$\rho(X) < \rho(X_p).$$

This is always true when $\rho(X)$ is odd, thanks to the Tate conjecture. Theorem 11.2 produces some new cases:

**Corollary 11.7.** Let $X$ be either a K3 surface of Picard rank 2 with $E$ a totally real field of degree 4 or a K3 surface of Picard rank 4 with $E$ a totally real field of even degree. Then $X$ contains infinitely many rational curves.

### 12 Lecture 12: Algorithms to compute the geometric Picard rank II: Sho Tanimoto (8/3)

In this lecture, we conclude the proof of Theorem 11.1.
12.1 Discriminants of Néron-Severi groups

Let $k$ be a number field and $X$ a K3 surface defined over $k$. Let $T_X$ be the transcendental rational Hodge structure and $E$ the endomorphism algebra of $T_X$.

**Proposition 12.1.** [6, Proposition 18] Assume that $E$ is a totally real field and that the dimension of $T_X$ over $E$ is odd. Then there exist infinitely many primes $p_i$ and infinitely many primes $q_j$ such that

1. $X$ has good, ordinary reduction at both $p_i$ and $q_j$;
2. $\rho(X_{p_i}) = \rho(X_{q_j}) = \rho(X) + [E : \mathbb{Q}]$;
3. square classes of $\text{NS}(X_{p_i})$ and $\text{NS}(X_{q_j})$ are different.

See Section 4 of [6].

12.2 An algorithm to compute the geometric Picard rank

Here is an algorithm to compute the geometric Picard rank: Let $X$ be a K3 surface defined over a number field $k$. Run the three following algorithms alongside each other:

1. Going through Hilbert schemes of a suitable projective space, find divisors on $X$ and compute the dimension of their span in the Néron-Severi group via intersection theory. This gives a lower bound for the Picard number.

2. Going through Hilbert schemes of a suitable projective space, find codimension 2 cycles in $X \times X$. Using intersection theory again, use these to get a lower bound on $[E : \mathbb{Q}]$ where $E$ is the endomorphism algebra of the transcendental Hodge structure.

3. Going through finite places $p$ of $k$, compute the Picard number and the discriminant of the Néron-Severi group of $X_{p_i}$ by counting points over finite fields and the Artin-Tate formula. This gives us an upper bound.

If the Hodge conjecture for $X \times X$ is true, then the upper bound and the lower bound must coincide. So the algorithm terminates in finite time. See Section 5 of [6].

13 Lecture 13: Kuga Satake construction I: Lars Halvard Halle (31/3, 7/4)

In this lecture and next lecture, we study Kuga-Satake construction which associates the abelian variety to a polarized Hodge structure of K3 type. We follow the exposition of [7] as well as [13, Chapter 4].
13.1 Hodge structures: revisit

Here we recall the definition of Hodge structures and its basis properties again. Let $S = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ be the Weil restriction of $\mathbb{G}_m$, $\mathbb{C}$ over $\mathbb{R}$. This is a real algebraic group defined over $\mathbb{R}$. There is a natural embedding $w : \mathbb{G}_m, \mathbb{R} \to S$ and we denote the inverse of norm by $t : S \to \mathbb{G}_m, \mathbb{R}$. Note that we have $t \circ w(x) = x^{-2}$. Let $V$ be a finite dimensional vector space over $\mathbb{Q}$. There are equivalent definitions of rational Hodge structures of weight $n$ on $V$:

- a representation $\rho : S \to \text{GL}(V_\mathbb{R})$ with $\rho w(x) \ast v = x^n \cdot v$;
- a decomposition $V_\mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$, $\overline{V^{p,q}} = V^{q,p}$.

Suppose that we have a representation $\rho : S \to \text{GL}(V_\mathbb{R})$. Then the diagonalization of $\rho(i)$ gives us the decomposition

$$V_\mathbb{C} = \bigoplus_{p+q=n} V^{p,q}.$$ 

Conversely, if we have such a decomposition then defines the operator $J : V_\mathbb{R} \to V_\mathbb{R}$ by acting on $V^{p,q}$ by $i^{p-q}$. This defines the representation $\rho : S \to \text{GL}(V_\mathbb{R})$.

Recall that $\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z}$ comes with the Hodge structure of the type $(-n, -n)$. A polarization of a Hodge structure $V_\mathbb{Z}$ of weight $n$ is a morphism of Hodge structures $\psi : V \otimes V \to \mathbb{Z}(-n)$, such that on $V_\mathbb{R}$, the real bilinear form

$$(2\pi i)^n \psi(x, Jy),$$

is symmetric and positive definite.

Recall for each an abelian variety $A$, one can associate a polarized integral Hodge structure $H^1(A, \mathbb{Z})$ and this gives an equivalence functor between the category of polarized abelian varieties and the category of polarized Hodge structures of weight 1.

Let $G$ be a reductive group over $\mathbb{Q}$, and $t, w$ are morphisms defined over $\mathbb{Q}$

$$\mathbb{G}_m \to^w G \to^t \mathbb{G}_m,$$

with $t \circ w(x) = x^{-2}$ and $w$ central. Moreover suppose that there is a morphism $h : S \to G_\mathbb{R}$ such that $t, w$ for $S$ and $G_\mathbb{R}$ are compatible.

A representation $V_\mathbb{Q}$ of $G$ is said to be homogeneous of weight $n$ if $w(x) \ast v = x^n \cdot v$ for any $v \in V$. Suppose that $V$ is a homogeneous representation of weight $n$. Then, $h$ equips $V$ with a rational Hodge structure of weight $n$. 

We consider \( Q(n) \) as a representation of \( G \) by
\[
g \ast v = t(g)^n \cdot v.
\]
A polarization of a representation \( V_Q \) of \( G \) of weight \( n \) is an \( G \)-invariant form
\[
\psi : V \otimes V \rightarrow Q(-n)
\]
such that \( (2\pi i)^n \psi(v, h(i)w) \) is a symmetric and positive definite form on \( V_{\mathbb{R}} \).

**Proposition 13.1.** [7, Proposition 2.11] The following conditions are equivalent:

1. \( \text{ad}(h(i)) \) is a Cartan involution of \( \text{Ker}(t)_{\mathbb{R}} \);
2. every homogeneous representation of \( G \) is polarizable;
3. \( G \) admits a faithful family of polarizable homogeneous representations;
   If \( G \) is connected, these conditions are further equivalent to
4. \( G \) admits a polarizable representation \( \rho \) such that \( \text{Ker}(\rho) \cap \text{Ker}(t) \) is finite.

### 13.2 Clifford algebra and Spin group

Here we recall the construction of Clifford algebras and Spin groups. For more details, see [13, Chapter 4] and references in it.

Let \( K \) be a commutative ring of characteristic zero, e.g., \( K = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \). Consider a free \( K \)-module \( V \) of finite rank, and a quadratic form \( q \) on \( V \) over \( K \). The associated bilinear form is defined by
\[
q(v, w) := \frac{1}{2}(q(v+w) - q(v) - q(w)),
\]
whose values are in \( K[1/2] \). The tensor algebra is defined by
\[
T(V) = \bigoplus_{i \geq 0} V^\otimes_i
\]
which is a graded non-commutative algebra over \( K \). Note that \( V^\otimes^0 = K \) by the definition. Let \( I(q) \) be the two sided ideal generated by elements \( v \otimes v - q(v) \) for any \( v \in V \) and we define the Clifford algebra:
\[
\text{Cl}(V) = \text{Cl}(V, q) = T(V)/I(q).
\]
Since generators for \( I(q) \) has even degree, the Clifford algebra has a \( \mathbb{Z}/2 \)-grading:
\[
\text{Cl}(V) = \text{Cl}(V)^+ \oplus \text{Cl}(V)^-.
\]
The algebra $\text{Cl}(V)^+$ is called the even Clifford algebra, and $\text{Cl}(V)^-$ is a two sided $\text{Cl}(V)^+$-ideal. By the construction, we have

$\mathbf{v} \cdot \mathbf{v} = q(\mathbf{v}), \quad \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} = 2q(\mathbf{v}, \mathbf{w}).$

In particular, when $\mathbf{v}, \mathbf{w}$ are orthogonal, we have $\mathbf{v} \cdot \mathbf{w} = -\mathbf{w} \cdot \mathbf{v}$.

Suppose that $K$ is a field and $v_1, \ldots, v_n$ an orthonormal basis for $V$. Then one can define a linear map mapping $v_{i_1} \cdot v_{i_2} \cdots v_{i_k} \mapsto \mathbf{v}_{i_1} \wedge \cdots \wedge \mathbf{v}_{i_k}$ and this defines an isomorphism of vector spaces:

$\text{Cl}(V) \cong \bigwedge^\ast V.$

In particular, we have $\dim \text{Cl}(V) = 2^n$.

From now on we assume that $K$ is a field. We denote the unit group of $\text{Cl}(V)$ by $\text{Cl}(V)^*$, and define the Clifford group by

$C\text{Spin}(V) = \{ \mathbf{v} \in \text{Cl}(V)^* \mid \mathbf{v}V\mathbf{v}^{-1} = V \}.$

One defines the even Clifford group $C\text{Spin}^+(V)$ in a similar way. One defines a morphism of $C\text{Spin}(V)$ into $O(V)$

$g \mapsto (v \mapsto gvg^{-1}).$

This gives us the exact sequence:

$0 \to \mathbb{G}_m \to ^w C\text{Spin}^+(V) \to O(V) \to 0.$

The Spin group is the subgroup:

$\text{Spin}(V) := \{ v \in C\text{Spin}^+(V) \mid v \cdot v^\ast = 1 \}$

where $v \mapsto v^\ast$ is the anti-automorphism of $\text{Cl}(V)$ mapping $v_1 \cdots v_k \mapsto v_k \cdots v_1$. This gives us the exact sequence

$0 \to \text{Spin}(V) \to \text{CSpin}^+(V) \to \mathbb{G}_m,$

such that $t \circ w(x) = x^{-2}$.

One denotes by $(\text{Cl}^+(V))_s$, the representation of $\text{CSpin}^+(V)$ on $(\text{Cl}^+(V))_s$ given by

$g \ast v = g \cdot v.$

One denote by $(\text{Cl}^+(V))_{ad}$ by the representation of $\text{CSpin}^+(V)$ via $O(V)$ given by

$g \ast v = g \cdot v \cdot g^{-1}.$

The action of $\text{CSpin}^+(V)$ on $(\text{Cl}^+(V))_s$ is compatible with the right $\text{Cl}^+(V)$-module, and one has the following isomorphisms:

$\text{End}_{\text{Cl}^+(V)}((\text{Cl}^+(V))_s) \cong (\text{Cl}^+(V))_{ad}.$
We have an isomorphism of representations:

\[(\text{Cl}^+(V))_{ad} \cong \bigoplus_{2i} \bigwedge^V.\]

The following lemma will be a key for next lecture:

**Lemma 13.2.** [7, Proposition 3.5] Let \( \Gamma \) be a Zariski dense subgroup of \( \text{Spin}(V) \) and \( \phi \) an automorphism of the \( K \)-algebra \( \text{Cl}^+(V) \) which commutes with the action of \( \Gamma \) by \( \gamma \mapsto (x \mapsto \gamma \cdot x \cdot \gamma^{-1}) \). Then \( \phi \) is the identity.

### 13.3 The construction of the abelian variety associated to a K3 surface

Let \((V_\mathbb{Z}, \psi)\) be a polarized integral Hodge structure of K3 type. Then we have a representation

\[ h : S \rightarrow \text{SO}(V_\mathbb{R}, \psi). \]

This naturally lifts to \( \tilde{h} : S \rightarrow \text{CSpin}^+(V_\mathbb{R}) \). (See [13, Chapter 4 Section 2.1] for an explicit construction of this map.)

**Theorem 13.3.** [7, Lemma 4.4 and Proposition 4.5] The Hodge structure for \((\text{Cl}^+(V))_{ad}(-1)\) is of K3 type. The Hodge structure for \((\text{Cl}^+(V))_s\) is of type \((1,0), (0,1)\).

**Theorem 13.4.** [7, Lemma 4.3] Relative to \( \tilde{h} \), any representation of \( \text{CSpin}^+(V_\mathbb{Q}) \) is polarizable.

**Proof.** Alternatively one can consult [13, Chapter 4 Proposition 2.4] for \((\text{Cl}^+(V))_s\). \(\Box\)

Thus we obtain an abelian variety

\[ A_V = \text{Cl}^+(V_\mathbb{R})/\text{Cl}^+(V_\mathbb{Z}) \]

associated to a polarized integral Hodge structure of K3 type \( V \).

**Proposition 13.5.** [13, Chapter 4 Proposition 2.5] There is an inclusion of Hodge structures of weight 2:

\[ V \hookrightarrow \text{Cl}^+(V) \otimes \text{Cl}^+(V). \]

### 14 Lecture 14: Kuga Satake construction II: Dan Petersen (12/4)

We continue the discussion of Kuga-Satake construction following [7].
14.1 Variations of Hodge structures

There is another equivalent definition of Hodge structures using Hodge filtrations. Suppose that we have a rational Hodge structure $V$ of weight $n$ with

$$V_C = \bigoplus_{p+q=n} V^{p,q}.$$ 

Then we define the Hodge filtration $F^\bullet V$ by

$$F^p V_C = \bigoplus_{r \geq p} V^{r,n-r},$$

which is a decreasing filtration satisfying

$$V_C = F^p V_C \oplus F^{n-p+1} V_C.$$ 

The Hodge decomposition is recovered by

$$V^{p,q} = F^p V_C \cap F^q V_C.$$ 

Suppose that we have a family $\pi : \mathcal{X} \to S$ of compact Kähler manifolds over a connected base. We have the local system $R^k \pi_* \mathbb{Z}$ whose stalk at $s \in S$ is isomorphic to $H^2(X_s, \mathbb{Z})$. Suppose that $S$ is simply connected so that the local system trivializes and isomorphic to

$$R^k \pi_* \mathbb{Z} \cong S \times H^k(X_0, \mathbb{Z}),$$

for some $0 \in S$. Then since Hodge numbers are constant in the family, we denote $b_{p,k} = \dim F^p H^k(X_0, \mathbb{C})$. We have a natural map

$$S \ni s \mapsto F^p H^k(X_s, \mathbb{C}) \in \text{Gr}(b_{p,k}, H^k(X_0, \mathbb{C})).$$

This is holomorphic (H1, Theorem 10.9)). Let $\mathcal{F}^{p,k}$ be the pullback of the universal bundle on $\text{Gr}(b_{p,k}, H^k(X_0, \mathbb{C}))$. This is a subbundle of the following holomorphic bundle:

$$\mathcal{H}^k = R^k \pi_* \mathcal{C} \otimes \mathcal{O}_S.$$ 

This is called the Hodge bundle, and $\mathcal{F}^{p,k}$ is an example of variations of Hodge structures. The category of variations of Hodge structures is closed under usual tensor operations, and one can define the notion of polarizations. Then we have the following equivalence:

**Theorem 14.1.** The functor $(\pi : \mathcal{A} \to S) \mapsto R^1 \pi_* \mathbb{Z}$ identifies the polarized abelian schemes over $S$ with the polarized variations of Hodge structures of type $\{(1,0), (0,1)\}$ on $S$. 
Suppose that we have a polarized Hodge structure \((V, \psi)\) of K3 type. Then consider the following domain

\[ D = \{ [\omega] \in \mathbb{P}(V_C) \mid \psi(\omega, \omega) = 0, \psi(\omega, \bar{\omega}) > 0 \}^+. \]

where + denotes one of connected components. Then one has the variation of polarized Hodge structures \(H\) of K3 type over \(D\).

Now suppose that \(W_\mathbb{Q}\) is a representation of weight \(n\) of the algebraic group \(\text{CSpin}^+(V_\mathbb{Q})\). Let \(W_\mathbb{Q}\) be the constant local system over \(D\). Then \(\text{CSpin}^+(V_\mathbb{Q})\) acts on \((D, W)\) by

\[ \gamma \ast (w \text{ based at } \omega \in D) = (\gamma w \text{ based at } \gamma \omega \gamma^{-1} \in D). \]

Each \(\omega \in D\) defines \(\tilde{h} : \text{SCSpin}^+(V_\mathbb{Q})\), and this gives us a variation of Hodge structures on \(W\). This construction is compatible with the tensor product. So a polarization \(\psi : W \otimes W \to \mathbb{Q}(-n)\) defines a polarization \(W\).

**Proposition 14.2.** \(^[7]\) Proposition 5.7] Let \((H, \psi)\) be a polarized variation of Hodge structure of K3 type over a connected smooth scheme \(S\). Then there exist

1. a finite stale surjective covering \(u : S_1 \to S\);
2. an abelian scheme \(\pi : A \to S_1\);
3. a \(\mathbb{Z}\)-algebra \(C\) and \(\mu : C \to \text{End}_{S_1}(A)\);
4. an isomorphism of variations of Hodge structure

\[ u : \text{Cl}^+(H, \psi) \cong \text{End}_C(R^1\pi_*\mathbb{Z}) \]

which induces an isomorphism of local systems of algebras

\[ u_\mathbb{Z} : \text{Cl}^+(H_\mathbb{Z}, \psi) \cong \text{End}_C(R^1\pi_*\mathbb{Z}). \]

**14.2 Kuga-Satake construction in families**

Suppose that we have a family \(\pi : X \to S\) of polarized K3 surfaces with a polarization \(\eta\). Then \(R^2\pi_*\mathbb{Z}(1)\) admits a variation of Hodge structure of K3 type. Let \(P^2\pi_*\mathbb{Z}\) be the orthogonal complement of \(\eta\). Then \(P^2\pi_*\mathbb{Z}\) also admits a natural polarized variation of Hodge structure of K3 type. For each \(s \in S\) we have a representation

\[ \pi(S, s) \to O(P^2(X_s, \mathbb{Z})(1), \psi). \]

**Proposition 14.3.** \(^[7]\) Proposition 6.4] For any polarized complex K3 surface \(X_0\), there exists a family of polarized K3 surface \(\pi : X \to S\) with \(X_s \cong X_0\) for some \(s \in S\), and such that the image of the above representation has finite index.
14.3 An Application to Weil conjecture for K3 surfaces

Using this, one can prove the following proposition:

**Proposition 14.4.** [2, Proposition 6.5] Let $X_0$ be a polarized K3 surface over a field $K$ of characteristic 0. Then there exist

1. a scheme $S$ of finite type over $\mathbb{Q}$, and a family $\pi : X \to S$ of polarized K3 surfaces;
2. a finite extension $K'$ of $K$ and $\nu : \text{Spec}(K') \to S$ such that $X_\nu$ is isomorphic to $X_0 \otimes_K K'$;
3. an abelian scheme $a : A \to S$;
4. a $\mathbb{Z}$-algebra and $\mu : C \to \text{End}_S(A)$
5. an isomorphism of $\mathbb{Z}_\ell$-sheaves of algebras over $S$;

$$h : \text{Cl}^+(P^2, \pi_\ast \mathbb{Z}_\ell(1), \psi_\ell) \cong \text{End}_C(R^1 a_\ast \mathbb{Z}_\ell).$$

14.3 An Application to Weil conjecture for K3 surfaces

The above proposition has the following application:

**Theorem 14.5.** [7, Theorem 1.3] Let $X$ be a K3 surface defined over a finite field which lifts to a K3 surface over a field of characteristic zero. Then $X$ satisfies the Weil conjecture.

This follows from Proposition 13.5 as well as Proposition 14.4. See [7, Section 6.6]. Any K3 surface over a finite field admits a lift to characteristic zero. We will discuss this in next lecture.

15 Lecture 15: Introduction to Crystalline cohomology:
Dustin Clausen (10/5)

15.1 Algebraic de Rham cohomology

Let $X$ be a smooth affine variety over $\mathbb{C}$. Then one can associate a complex manifold $X(\mathbb{C})$ and its de Rham complex:

$$\text{dR}(X(\mathbb{C}); \mathbb{C}) = (\mathcal{A}^0_{c,\infty}(X) \to d \mathcal{A}^1_{c,\infty}(X) \to d \mathcal{A}^2_{c,\infty}(X) \to \cdots)$$

Then this complex contains a subcomplex: the algebraic de Rham complex:

$$\text{AdR}(X(\mathbb{C}); \mathbb{C}) = (\Omega^0_{\text{alg}}(X) \to d \Omega^1_{\text{alg}}(X) \to d \Omega^2_{\text{alg}}(X) \to \cdots)$$

**Theorem 15.1** (Grothendieck). *This inclusion is quas-isomorphic.*
Remark  The variety $X$ needs to be affine for this theorem, however, there is a version of this theorem for arbitrary scheme $X$. It is more complicated to state although it is easy to prove using hyper cohomology of complexes of sheaves of differentials and GAGA.

Example  Consider the following a punctured elliptic curve:

$$X = \text{Spec}(\mathbb{C}[x, y]/(y^2 - f(x))) =: \text{Spec}(A),$$

where $f$ is a cubic polynomial with distinct roots. Then we have

$$\Omega^0_X = A, \quad \Omega^1_X = A(dx, dy)/(2ydy - f'(x)dx), \quad \Omega^2_X = 0.$$ 

The module $\Omega^1_X$ is free of rank 1 generated by $\omega = dx \frac{dy}{y} = dy f'(x)$. We have the differential $d : \Omega^0_X \to \Omega^1_X$. Then $\ker(d) = \mathbb{C}$ and $\text{coker}(d) = \mathbb{C} \omega \oplus \mathbb{C} x \omega$. Thus we have

$$H^i_{dR}(E \setminus \{0\}) = \begin{cases} \mathbb{C} & i = 0 \\ \mathbb{C} \oplus \mathbb{C} & i = 1 \end{cases}.$$ 

15.2 Algebraic de Rham cohomology in families

Suppose that we have a smooth proper family $p : X \to S$. Then we have a relative de Rham complexes

$$\text{AdR}(X/S; \mathbb{C}) = (\Omega^0_{X/S} \to^d \Omega^1_{X/S} \to^d \Omega^2_{X/S} \to \cdots)$$

The associated cohomology will be locally free sheaves on $S$. But there is more structure... Where does it come from?

Recall that

$$d\text{R-coh} = \text{sing-coh},$$

and the singular cohomology also have a relative notion as local systems. The local systems are rigid in the sense that if we take a contractible neighborhood, then the local system becomes a constant system. We should be able to see this structure for de Rham cohomology.

One can ask if there is a purely algebraic way to recover

$$\text{AdR}^*(X/S)|_{\hat{s}_0}$$

from $\text{AdR}^*(X_0/s_0)$? Here $\hat{s}_0$ is the formal neighborhood of $s_0$. In other words, one can ask whether $\text{AdR}^*(X/S)$ is crystalline.

Example  Let $f_\lambda(x) = x(x-1)(x-\lambda)$ where $\lambda \in S = \mathbb{C} \setminus \{0, 1\}$. Let

$$\mathcal{E}_\lambda = \text{Spec}(\mathbb{C}[x, y, \lambda]/(y^2 - f_\lambda(x)))$$
15.2 Algebraic de Rham cohomology in families

which defines a family of the punctured elliptic curves. Recall that we have a basis for $H^1_{\text{dR}}(E_{/S})$

$$\omega, x\omega$$

Since $f_\lambda(x)$ and $f'_\lambda(x)$ are coprime, one can find polynomials $A, B$ in $x$ whose coefficients are rational functions in $\lambda$ such that

$$Af_\lambda(x) + Bf'_\lambda(x) = 1.$$ 

Then $\omega$ is given by

$$\frac{1}{2}Aydx + Bf'_\lambda(x)dy.$$ Then $\frac{d}{d\lambda}\omega$ and $\frac{d}{d\lambda}(x\omega)$ can be expressed as linear combinations of $\omega, x\omega$ such that coefficients are rational functions. We have the relation

$$2ydy = f'_\lambda(x)dx$$

and

$$dx = y\omega, \quad dy = \frac{1}{2}f'_\lambda(x)\omega.$$ 

But in the full module $\Omega^1_{E/S}$, the relation lifts to

$$2ydy = f'_\lambda(x)dx + \frac{df_\lambda}{d\lambda}d\lambda.$$ 

Thus we obtain

$$dx \wedge d\lambda = y\omega \wedge d\lambda,$$

$$dy \wedge d\lambda = \frac{1}{2}f'_\lambda(x)\omega \wedge d\lambda,$$

$$dx \wedge dy = \frac{1}{2}d\lambda \omega \wedge d\lambda.$$ 

Thus we have

$$d\omega = Ady \wedge dx + A\lambda y d\lambda \wedge dx + 2B_x dx \wedge dy + 2B_\lambda d\lambda \wedge dy$$

$$= (B_x \frac{df_\lambda}{d\lambda} - \frac{1}{2}A \frac{df_\lambda}{d\lambda} - A\lambda y - B_\lambda f'_\lambda(x))\omega \wedge d\lambda.$$ 

Thus we have

$$\frac{d}{d\lambda} \omega = (B_x \frac{df_\lambda}{d\lambda} - \frac{1}{2}A \frac{df_\lambda}{d\lambda} - A\lambda y - B_\lambda f'_\lambda(x))\omega.$$ 

Now we would like to discuss a number theoretic prediction. There is analogy between

$$\mathbb{Z}_p, \quad \mathbb{C}[\![t]\!].$$ 

If we have a smooth proper morphism $f : X \to \text{Spec}(\mathbb{Z}_p)$, then one can define the algebraic de Rham cohomology:

$$H^*_{\text{dR}}(X \otimes \mathbb{Q}_p/\mathbb{Q}_p)$$
There should be secretly a functor

\[ \Phi : \{ \text{smooth proper varieties over } \mathbb{F}_p \} \to \{ \text{finite dimensional vector spaces over } \mathbb{Q}_p \} \]

such that if \( X/\mathbb{F}_p \) lifts to \( X/\mathbb{Z}_p \), then the functor gives \( H^*_\text{dR}(X \otimes \mathbb{Q}_p/\mathbb{Q}_p) \). In particular there should be Frobenius action on \( H^*_\text{dR}(X \otimes \mathbb{Q}_p/\mathbb{Q}_p) \).

16 Lecture 16: The lifting property of K3 surfaces in positive characteristic: TBA (TBA)

References

[1] W. L. Baily and A. Borel, Compactification of arithmetic quotients of bounded symmetric domains, 7.4, 7.5, 8.3

[2] W. P. Barth, K. Hulek, C. A.M. Peters, and A. Van de ven, Compact Complex Surfaces, (document) 1.1.1, 2.1, 3.1, 4.1, 4.1, 6

[3] A. Beauville, Complex Algebraic Surfaces,


[5] R. E. Borcherds, Automorphic forms on \( O_{s+2,2}(\mathbb{R}) \) and infinite products, 8.4


[8] J. S. Ellenberg, K3 surfaces over number fields with geometric Picard number one, 9.1, 9.1

[9] V. A. Gritsenko, K. Hulek, and G. K. Sankaran, The Kodaira dimension of the moduli of K3 surfaces, 8.2, 8.3, 8.4, 8.4

[10] R. Hartshorne, Algebraic geometry, 1.1.1


[12] B. Hassett and Y. Tschinkel, Rational points on K3 surfaces and derived equivalences, 4.10
REFERENCES

[13] D. Huybrechts, *Lectures on K3 surfaces*, 1, 2, 2.1, 3, 5.12, 6, 6.3, 6.4, 6.8, 6.9, 6.10, 7, 7.2, 7.3, 7.6, 7.8, 8.1, 13, 13.2, 13.3, 13.6, 13.8, 13.9, 13.10


[21] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, 4.6, 4.7

[22] K. G. O'Grady *Compact hyperkähler manifolds: an introduction*, 6.3

[23] B. Poonen, D. Testa, and R. van Luijk, *Computing Néron-Severi groups and cycle class groups*, 11


[26] S. G. Tankeev, *Surfaces of K3 type over number fields and the Mumford-Tate conjecture*, 11.1

[27] S. G. Tankeev, *Surfaces of K3 type over number fields and the Mumford-Tate conjecture II*, 11.1


[29] R. van Luijk, *K3 surfaces with Picard number one and infinitely many rational points*, 10, 10.1, 10.4, 2, 10.3

[31] C. Voisin, *Hodge theory and complex algebraic geometry, I*, 5.2 5.7 5.8 5.2 14.1


Department of Mathematical Sciences
University of Copenhagen
Universitetspark 5
2100 Copenhagen Ø
Denmark

sho@math.ku.dk
http://shotanimoto.wordpress.com